

Computer Algebra Application of the Bondi-Sachs Formalism to Scalar-Tensor Gravity

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In loving memory of Helmuth

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Abstract

The Bondi-Sachs formalism describes the flow of energy radiated away from a finite, isolated gravitational system in the form of gravitational waves. This flow results in a change in the mass of the system, expressed through the so-called news function in the Bondi mass-loss formula. Moreover, consideration of the asymptotic symmetries of the Bondi-Sachs metric yields a symmetry group larger than the expected Poincaré group, known as the Bondi Metzner Sachs (BMS) group. An additional prediction of the formalism is the existence of a gravitational wave memory effect.

The goal of this work is to develop a software package for the Computer Algebra System (CAS) Maxima, for the assisted application of the Bondi-Sachs formalism to alternative theories of gravity.

A brief introductory survey of the Bondi-Sachs formalism is presented, together with a summary of the existing application of this framework to the Brans-Dicke theory, a particular case of scalar-tensor theory of gravity.

The developed software package is used to produce novel results, by applying the framework to the Bocharova-Bronnikov-Melnikov-Bekenstein theory, a conformal scalar field theory, in axial symmetry.

This work closes with a list of possible issues encountered while employing the conventional algorithm to solve the field equations of the Bondi metric, which in some cases may result in a theory having no asymptotically flat spacetime where the scalar field is non-trivial.

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Notation and Conventions

In this work, we have the challenge of working with two different perspectives, each of which benefit from different notations. Instead of choosing one over the other, we make use of the notations best suited for each case. In particular, when dealing with abstract results, valid independent of the choice of coordinate system, we adapt the abstract index notation as described by Wald in his introductory text of general relativity [41] to our purposes. In particular:

- $a, b, c, \dots = 0, 1, 2, 3$ denote the components of objects in 4 dimensional manifolds.
- $A, B, C, \dots = 2, 3$ denote the components of objects in the round 2-sphere submanifold spanned by two angular-like components.
- g_{ab} is the a space-time metric.
- h_{AB} is a conformal 2-metric.
- q_{AB} is the unit sphere metric.
- ∇_a is the metric compatible derivative operator in space-time, i.e. Levi-Civita connection.
- D_A is connection compatible with the conformal 2-metric h_{AB} .
- $\bar{\partial}_A$ is connection compatible with the unit sphere metric.

On the other hand, as a consequence of the current limitations of the developed software, concrete results in this thesis are usually written in a axial symmetric system of coordinates. Such expressions include multiple partial derivatives and benefit from a more succinct notation. Thus, we denote partial derivatives by subindices, similar to the notation used in text book by Courant and Hilbert [13]: for a some function f , we express its partial derivatives as f_u , f_r , f_θ and f_ϕ . However, we remark—to avoid confusion—that each time we are working with such results, we avoid using letter subindices as dummy indices, and reserve them to denote exclusively partial derivatives instead.

The per chapter structure used in this thesis consists of a general introduction to the specific theory, for which an abstract notation is best suited, and a transition to component notation at the end of the chapter, where results like mass-loss formulae are calculated using a computer algebra system.

For the metric, we use throughout, and without exception, the $(-, +, +, +)$ metric signature. Results taken from literature using different signature conventions, such as the original work of Bondi, are consequently converted.

We choose to work with geometrized units, i.e.

$$G = c = 1.$$

Thus, Einstein's equation takes the form

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}$$

Points that introduce changes to the notation and convention are respectively marked.

To aid in readability, we introduce the following abbreviations, which are also detailed throughout the text:

CAS	Computer Algebra System
GR	General Relativity
BD	Brans-Dicke theory
BBMB	Bocharova-Bronnikov-Melnikov-Bekenstein theory
EGBD	Einstein-Gauss-Bonnet theory

Chapter 1

Introduction

In his 1960 publication [9], Hermann Bondi introduced a particular spacetime metric to study the emission of gravitational waves from a finite, isolated, and axially symmetric material system. His later 1962 publication in collaboration with van der Burg and Metzner [10] explains the relation between the emission of said gravitational waves and the change in the mass of the system, as described by the so-called news function. Furthermore, this result would show that the structure of the asymptotic symmetry group of General Relativity is in fact larger than the Poincaré group—including a continuous group of transformations known today as the Bondi-Metzner-Sachs (BMS) group.

Simultaneous work by Rainer K. Sachs [33, 32] would extend this formalism to a more general metric, now known as the Bondi-Sachs metric, which no longer requires any particular symmetry of the system.

1.1 The Bondi-Sachs Formalism

The Bondi-Sachs formalism studies isolated gravitational systems through a metric-based treatment of Einstein's equation. It describes the flow of energy radiated away from the system along null rays in the form of gravitational waves, as observed at future null infinity, and explains how the change in mass of the system, expressed through the so-called mass-loss formula, is an effect of the non-linearity of the equations. It also shows that, different to what one would expect, the symmetry group at future null infinity is not given by the Poincaré group, but rather a larger extension of it known as the BMS group. Another curious result of the theory is the prediction of gravitational memory effects—permanent relative displacement between free falling particles due to a burst of gravitational waves [34, 24].

The relatively recent experimental measurement of gravitational waves [12], and the following development of newer detectors (see for example [16]), covering an increasing gravitational spectrum, has sparked renewed interest in the Bondi-Sachs formalism, both in its extensions to alternative theories of gravity, as to its possible applications in the search for a quantum theory of gravity [24].

1.2 Goals of this Thesis

Other groups have already studied the application of the Bondi-Sachs formalism to the Brans-Dicke theory, a case of scalar-tensor gravity [36, 20]. This work’s goal is to extend the Bondi-Sachs formalism to, possibly many, other alternative theories of gravity through the automatization of the required calculations. For this, the algorithmic nature of solving Einstein equations with the Bondi metric is used. Calculations are carried out using Maxima [26]—a free-software computer algebra system¹. The resulting source code is intended to be made available, such that similar results may be computed for other theories with minimal manual intervention. This reduces the, somewhat redundant, effort required to calculate new mass-loss formulae or to solve for the parts of the Bondi metric when studying alternative theories of gravity. However, currently, Maxima does not provide any mature means to work with tensorial objects in a coordinate independent way. Therefore, results derived with help from software are limited to a particular choice of coordinate system. Future work is planned to develop a tensor package for Maxima that is comparable in quality to the ones available in other platforms, e.g. xAct in Wolfram Mathematica.

Though software development comprised a significant portion of the work, the results presented in this text are independent from the software, and of physical interest by themselves. Their derivation is not beyond the capabilities of someone to either understand or reproduce. Thus, the discussion is also not centered in the software. Moreover, though this thesis focuses mainly in the calculation of mass-loss formulae, it shall become clear to the reader that the steps taken into producing said formulae serve also as a starting point to produce other results of the theory, such as, for example, the aforementioned gravitational wave memory effect and the symmetry groups of the theory at future null infinity.

1.3 Structure of this Thesis

This thesis tries to be as self-contained as possible. The reader is not assumed to have much previous knowledge about the Bondi-Sachs formalism.

Chapter 2 provides an introduction to the topic. It starts by presenting core ideas behind the Bondi-Sachs metric and the usefulness of the so-called Bondi gauge. It then continues with an explanation of the procedure to solve the equations of motion by expanding metric’s functions in negative powers of r , a light-like radial coordinate, with all other coordinates as arguments of the coefficients. Here, the equations known as supplementary conditions yield the mass-loss formula, a result that describes the energy lost by the system in the form of gravitational waves. Though much of the explanation is presented using a general metric, independent of the possible symmetries of the system, the chapter also introduces the Bondi coordinates, used for results calculated with assistance of Maxima, and explains the connection of the mass loss formula to the actual mass of the system.

¹This point is extremely important for the author. He refuses to publish work where the reproducibility is locked behind the paywall of a privative platform; thus, being of little use for the general public. Also, as a strong advocate of the four fundamental software freedoms—as defined by the free software foundation—he considers it unwise to make one’s workflow dependent on privative software.

A brief summary of the motivation behind scalar-tensor theories of gravity is provided in Chapter 3. The action of the most general scalar-tensor theory yielding second order equations of motion—Horndeski theory [19]—is given together with the special case of the Bergmann-Wagoner theory [4] that is studied in this thesis. Important concepts, like conformal frames and conformal transformation, are explained due to their use in later chapters.

The derivation of the Bondi mass-loss formula for Brans-Dicke theory is shown in Chapter 4. This result is already known in the literature; however, it was only somewhat recently published, and it provides a useful point of reference for the software developed in this work. Additionally, it introduces the discussion of asymptotic requirements for a scalar-tensor theory, and the extra evolution equations due to the coupled scalar field.

Chapter 5 discusses Bocharova-Bronnikov-Melnikov-Bekenstein theory [8, 3], and presents the first new result of this thesis. The specifics of the theory’s equations of motion yield, as discussed in the chapter, an ideal case for the software to compute a mass-loss formula.

The possible difficulties one may encounter while looking for mass-loss formulae are addressed in Chapter 6. First, the special case of Starobinsky gravity [35] is considered. The failure to find a mass-loss formulae is analyzed through the equation of the scalar field in a systematic algorithmic way. The chapter continues by extending this analysis to more general theories, differentiated by their potentials. Finally, complications, which not necessarily result in a lack of mass-loss formulae, are discussed through the theory of the Einstein-Gauss-Bonnet-dilaton [6].

This work concludes with an outlook into the different paths that possible continuations of this work may follow. This includes ideas like developing a more general, coordinate independent, version of the code, but also the experimental predictions that may follow from solutions the Bondi-Sachs metric, and mass-loss formulae in the context of alternatives theories of gravity.

1.4 On the Writing Style of this Work

As the nature of a thesis may suggest, this text is the product of a single individual. Nonetheless, the author has decided to write in first person plural for the following reasons: first, many statements originated from conversations with the author’s supervisor, and the plural writing is a way to provide some credit for this cooperation. Second, it is the author’s opinion that formulating the discussion in a way that treats the exposition as a dialog with the reader makes the text easier to read from a didactically point of view.

1.5 Access to the Source Code and Errata

As explained above, the software package developed together with this thesis is intended to be publicly available. The development is carried out on the author’s personal git repository, and has been made available under

<https://git.thomaslabs.org/bondi-sachs/>.

Similarly, a copy of this text, and possible future errata are available on the author's personal website

<https://thomaslabs.org/research/master-thesis.html>.

1.6 On the Topic of AI

Due to the increasing impact of the topic of AI in society and in sciences as a whole, it was considered important to make the following statement:

No Large Language Models (LLM) or any other set of similar programs, colloquially known as AI, were used while writing the software used to calculate the results presented in this thesis; or for writing any portion of text. The following document—for all its merits and faults—is the exclusive result of the work of humans.

Though the use of local, open weight models may have been deemed acceptable for the author, the quality of code generated by LLMs when the work started did not justify the increase in bureaucracy to publish an AI assisted thesis. However, the landscape of AI is changing fast, and this may no longer be the case. Just like the lean language has been shown to be extremely useful as a tool for LLMs to work in a more rigorous way, the present code could also be used in combination with LLMs to further the scope of the results. Should this be of interest, the reader is reminded of the importance—specially as scientists—of choosing open platforms, respectful of users digital rights, instead of supporting close services that actively exploit their users and harm society.

Chapter 2

Bondi-Sachs Formalism in General Relativity

Before embarking in the goal of this thesis, we consider it as helpful to provide a review of this research topic for readers unfamiliar with the original work of Bondi [10], or existing literature on this topic [25, 24, 2]. Readers already familiar with the formalism are still recommended to read this chapter, as it covers topics relevant for this text.

As commented in the introductory Chapter 1, the Bondi-Sachs formalism provides a metric based treatment of Einstein's equations and is built on coordinates constructed from a family of lightlike congruences. It describes the change in mass of an isolated gravitational system as an effect of the non-linear nature of Einstein's equations and measures the flow of energy leaving at future null infinity.

By future null infinity, we refer to the conformal compactification of future null infinity in the formalism introduced by Ted Newman and Roger Penrose [28, 29, 25], denoted in the literature by $\mathcal{I}^+$.

In Section 2.1, we present the construction of the Bondi-Sachs metric from a family of congruence of lightlike geodesics. Afterwards, in Section 2.2, we introduce the concepts of the main and evolution equations in the context of the Bondi-Sachs formalism, used to solve Einstein's equations by means of a series expansion in negative powers of a lightlike radial coordinate. Furthermore, the solution for the equations of motion in General Relativity is presented. Section 2.3 applies the ideas of the Bondi-Sachs formalism to classical electrodynamics, with the goal to motivate the idea that the Bondi mass-loss formula is a result of the non-linear nature of Einstein's equations. Due to the current limitations of the developed software package, a great portion of our results are restricted to an axially symmetric metric; Section 2.4 introduces the details of such special case. Finally, this chapter is closed in Section 2.5 with a transformation of the Weyl metric to the axi-symmetric Bondi metric. This shows the connection of some of the coefficients inside the Bondi metric with physical observables.

2.1 Bondi-Sachs Metric

The general Bondi-Sachs coordinates $x^a = (u, r, x^A)$ are constructed from a congruence of lightlike geodesics (rays) that are normal to the hypersurfaces $u = \text{const.}$, for some scalar field $u(x^a)$, along which wavefronts can travel. The normal vectors k^a are chosen to be future pointing and satisfy

$$k_a = -\partial_a u, \quad g^{ab} k_a k_b = 0. \quad (2.1)$$

Degenerate cases are eliminated by imposing the requirements

$$\rho = \frac{1}{2} \nabla_a k^a \neq 0, \quad (2.2)$$

$$|\sigma| = \frac{1}{2} (\nabla^a k^b)(\nabla_a k_b) - \rho^2 \neq \rho^2, \quad (2.3)$$

which respectively control how the light cone is enlarged and distorted.

From (2.1) follows immediately for the inverse metric $g^{uu} = 0$. The angular coordinates x^A are chosen to be constant along the null rays, i.e. $k^a \nabla_a x^A = 0$, hence $g^{uA} = 0$. Finally, the r coordinate, which varies along the null rays, is chosen to be an areal coordinate such that $\det g_{AB} = r^4 \det q_{AB}$, where q_{AB} is the standard unit 2-sphere metric, e.g. $q_{AB} dx^A dx^B = d\theta^2 + \sin^2 \theta d\phi^2$ for standard spherical coordinates. In the literature, this last requirement is called the Bondi gauge.

The definition of the inverse, $g^{ac} g_{cb} = \delta^a_b$, implies $g_{rr} = 0$ and $g_{rA} = 0$ from $0 = \delta^u_r = g^{ur} g_{rr}$ and $0 = \delta^u_A = g^{ur} g_{rA}$ respectively.

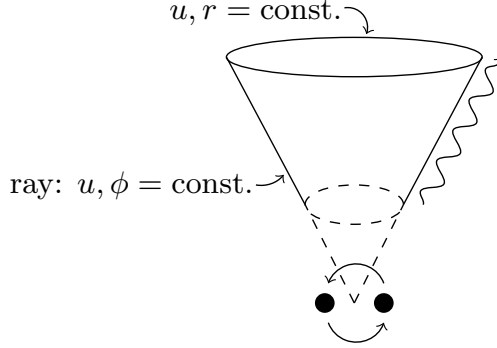


Figure 2.1: Illustration of the coordinate system with θ excluded.

From these considerations, the general Bondi-Sachs metric is given by

$$g_{ab} dx^a dx^b = -\frac{V}{r} e^{2\beta} du^2 - 2e^{2\beta} du dr + r^2 h_{AB} (d\sigma^A - U^A du)(d\sigma^B - U^B du), \quad (2.4)$$

where h_{AB} is a conformal 2-metric with two degrees of freedom and satisfying

$$g_{AB} = r^2 h_{AB} \quad \text{and} \quad \det h_{AB} = \det q_{AB}. \quad (2.5)$$

Thus,

$$\partial_r (\det h_{AB}) = 0. \quad (2.6)$$

Furthermore, from the identity for the derivative of the determinant

$$\frac{d \det A(t)}{dt} = \det(A) \operatorname{tr} \left(A^{-1} \frac{dA}{dt} \right), \quad (2.7)$$

and the non-degeneracy of h , we have

$$h^{AB} \partial_r h_{AB} = \det(h_{AB})^{-1} \partial_r (\det h_{AB}) = 0. \quad (2.8)$$

Similarly, we can also derive

$$h^{AB} \partial_u h_{AB} = 0, \quad h_{AB} \partial_u h^{AB} = 0 \quad \text{and} \quad h_{AB} \partial_r h^{AB} = 0. \quad (2.9)$$

Other components of the inverse metric follow directly from using, once again, $g^{ac} g_{cb} = \delta^a_b$. Its only non-zero components are:

$$g^{ur} = -e^{-2\beta}, \quad g^{rr} = \frac{V}{r} e^{-2\beta}, \quad g^{rA} = -U^A e^{-2\beta} \quad \text{and} \quad g^{AB} = \frac{1}{r^2} h^{AB}. \quad (2.10)$$

2.2 Equations of Motion

As stated in the introduction of this chapter, we intend to study the energy radiated by an isolated gravitational system as measured at future null infinity. In the case of General Relativity, the equations of motion are given by Einstein's equation:

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}. \quad (2.11)$$

Outside the finite material system of interest, it would be possible to assume a vanishing stress energy tensor, i.e. $T_{ab} = 0$. However, when considering more general theories of gravity, one often encounters fields which also couple to the gravitational field, either through additional dynamical terms or through potentials. This results in the introduction of an effective stress energy tensor

$$T_{ab}^{\text{eff}} = T_{ab}^{\text{M}} + T_{ab}^{\text{T}},$$

where the T_{ab}^{M} is related to the matter fields and T_{ab}^{T} encodes the contributions from additional fields of the theory. Therefore, we instead require only that the resulting stress-energy tensor be symmetric and divergence free, i.e.

$$T_{ab}^{\text{eff}} = T_{ba}^{\text{eff}} \quad \text{and} \quad \nabla^a T_{ab}^{\text{eff}} = 0. \quad (2.12)$$

Einstein's equations can be encoded by defining a tensor E_{ab} given by

$$E_{ab} = R_{ab} - \frac{1}{2} R g_{ab} - 8\pi T_{ab}^{\text{eff}}, \quad (2.13)$$

where R_{ab} is the Ricci curvature tensor and R is the scalar curvature, and requiring $E_{ab} = 0$. However, we first consider E_{ab} in general, without the latter requirement.

From the Bianchi identities

$$\nabla_a R^a_b = \frac{1}{2} \nabla_b R,$$

and Equation (2.12), we have

$$\nabla_a E_b^a = \partial_a E_b^a + \Gamma_{ac}^a E_b^c - \Gamma_{ab}^c E_c^a = 0. \quad (2.14)$$

The last term (2.14) can be rewritten by using the metric compatibility of the connection

$$\partial_a g^{bc} = -\Gamma_{ad}^b g^{dc} - \Gamma_{ad}^c g^{bd}$$

and the symmetry of E_{ab} , from which

$$(\partial_a g^{bc}) E_{bc} = -2\Gamma_{ad}^b E_b^d \quad (2.15)$$

follows. Hence, the divergence of E_{ab} can be expressed as

$$\nabla_b E_a^b = \frac{1}{\sqrt{|\det g|}} \partial_b \left(\sqrt{|\det g|} E_a^b \right) + \frac{1}{2} (\partial_a g^{bc}) E_{bc} = 0. \quad (2.16)$$

Let us now introduce the following equations:

$$E_a^u = 0 \quad \text{and} \quad E_{AB} - \frac{1}{2} g_{AB} g^{CD} E_{CD} = 0. \quad (2.17)$$

These are known as main equations in the literature about the Bondi-Sachs formalism. It can be directly checked that from their validity

$$\begin{aligned} E_r^b &= g_{ra} E^{ab} = g_{ur} E^{ub} = g_{ur} g^{bc} E_c^u = 0, \\ E_{ur} &= g_{ua} E_r^a = 0, \\ E_{rr} &= g_{ra} E_r^a = 0 \end{aligned}$$

and

$$E_B^A = \frac{1}{2} g^{CD} E_{CD} \delta^A_B \quad (2.18)$$

follow. Furthermore, recalling that $g^{uu} = 0$, we have for $a = r$ that Equation (2.16) reduces to

$$\begin{aligned} (\partial_r g^{AB}) E_{AB} &= -\frac{2}{r} g^{AB} E_{AB} + \frac{1}{2} (\partial_r h^{AB}) h_{AB} g^{CD} E_{CD} \\ &= -\frac{2}{r} g^{AB} E_{AB} \\ &= 0. \end{aligned}$$

Hence,

$$g^{AB} E_{AB} = 0, \quad (2.19)$$

which further implies

$$E_B^A = 0, \quad E_{AB} = \frac{1}{2} g_{AB} g^{CD} E_{CD} = 0, \quad \text{and} \quad (\partial_a g^{bc}) E_{bc} = 0. \quad (2.20)$$

Finally, due to $E_u^u = 0$ and $E_u^A = 0$, we see that the retarded time u and angular components x^A of Equation (2.16) reduce to

$$\partial_r (r^2 e^{2\beta} E_u^r) = 0 \quad \text{and} \quad \partial_r (r^2 e^{2\beta} E_A^r) = 0, \quad (2.21)$$

which are known as supplementary conditions.

The important observation of the theory is that, if the main equations (2.17) hold and the supplementary conditions (2.21) are fulfilled for some value of r , e.g. either on some finite worldtube or in the limit $r \rightarrow \infty$, then the latter, by nature of the resulting initial value problem, are also fulfilled for every other value of r . This can also be expressed by the fact that their solutions must be of the form

$$E_u^r = f(u, x^A) r^{-2} \quad \text{and} \quad E_u^r = g(u, x^A) r^{-2}, \quad (2.22)$$

for some arbitrary functions f and g .

We hint again at the important fact that at no point did we make any requirements about the theory other than $E_{ab} = E_{ba}$ and T_{ab}^{eff} being divergence free.

The main equations (2.17) can be further subdivided. First, into the hyper-surface equations

$$E_a^u = 0, \quad (2.23)$$

given in their expanded form by

$$E_r^u = 0 : \quad \partial_r \beta = \frac{r}{16} h^{AC} h^{BD} (\partial_r h_{AB}) (\partial_r h_{CD}) + 2\pi r T_{rr}, \quad (2.24a)$$

$$E_A^u = 0 : \quad \partial_r [r^4 e^{-2\beta} h_{AB} (\partial_r U^B)] = 2r^4 \partial_r \left(\frac{1}{r^2} D_A \beta \right) - r^2 h^{EF} D_E (\partial_r h_{AF}) + 16\pi r^2 T_{rA}, \quad (2.24b)$$

$$E_u^u = 0 : \quad 2e^{-2\beta} (\partial_r V) = \mathcal{R} - 2h^{AB} [D_A D_B \beta + (D_A \beta)(D_B \beta)] + \frac{e^{-2\beta}}{r^2} D_A [\partial_r r^4 U^A] - \frac{1}{2} r^4 e^{-4\beta} h_{AB} (\partial_r U^A) (\partial_r U^B) + 8\pi [h^{AB} T_{AB} - r^2 T_a^a], \quad (2.24c)$$

where D_A is the covariant derivative and $\mathcal{R}$ is the Ricci scalar with respect to the conformal 2-metric h_{AB} . The second group is called the evolution equations, given by

$$E_{AB} - \frac{1}{2} g_{AB} g^{CD} E_{CD} = 0, \quad (2.25)$$

which can be expanded by introducing a complex polarization dyad m^a with $m^a \nabla_a u = 0$, i.e. tangent to the null hypersurfaces and pointing in the angular direction. While the exact details of this expansion are beyond the objectives of this section (see instead [25]), we use the final result given by

$$m^A m^B \left\{ r \partial_r [r (\partial_u h_{AB})] - \frac{1}{2} \partial_r [r V (\partial_r h_{AB})] - 2e^\beta D_A D_B e^\beta + h_{CA} [\partial_r (r^2 U^C)]_B - \frac{1}{2} r^4 e^{-2\beta} h_{AC} h_{BD} (\partial_r U^C) (\partial_r U^D) + \frac{r^2}{2} h_{AB} (D_C U^C) + r^2 U^C D_C (\partial_r h_{AB}) - r^2 (\partial_r h_{AC}) h_{BE} (D^C U^E - D^E U^C) - 8\pi e^{2\beta} T_{AB} \right\} = 0. \quad (2.26)$$

A necessary, though by no means sufficient, condition for the Bondi-Sachs metric to describe an asymptotically flat spacetime is for it to approach the Minkowski metric

$$\eta_{ab} dx^a dx^b = -du^2 - 2dr du + r^2 q_{AB} dx^A dx^B \quad (2.27)$$

at future null infinity. To solve the equations of motion, and to ensure the above limit, we introduce for the functions of the metric (2.4), a power series ansatz in r^{-1} with the asymptotic behavior

$$\lim_{r \rightarrow \infty} \beta = \lim_{r \rightarrow \infty} U^A = 0, \quad \lim_{r \rightarrow \infty} r^{-1} V = 1 \quad \text{and} \quad \lim_{r \rightarrow \infty} h_{AB} = q_{AB}. \quad (2.28)$$

Thus, the concrete form of this expansion is given by

$$\beta(u, r, x^a) = \beta_1(u, x^A) r^{-1} + \mathcal{O}(r^{-2}) \quad (2.29a)$$

$$U^A(u, r, x^a) = U_1^A(u, x^A) r^{-1} + \mathcal{O}(r^{-2}) \quad (2.29b)$$

$$V(u, r, x^a) = r - 2M(u, x^a) + V_1(u, x^A) r^{-1} + \mathcal{O}(r^{-2}) \quad (2.29c)$$

$$h_{AB}(u, r, x^C) = q_{AB} + c_{AB}(u, x^C) r^{-1} + (h_2)_{AB}(u, x^C) r^{-2} + \mathcal{O}(r^{-3}). \quad (2.29d)$$

The functions M and c_{AB} are integration functions of the main equations, i.e. are not determined by them. They also have a special meaning deserving of additional names, as we shall see throughout the rest of this chapter.

In the case of GR, outside the bulk we take $T_{ab} = 0$. Now, suppose we have the value of h_{AB} for some $u = u_0$, the main equations can then be solved by the following algorithm:

1. Use h_{AB} to solve (2.24a) for β up to a integration constant. The latter can be showed to be removable through a suitable coordinate transformation [10].
2. With h_{AB} and β , solve for U^A in Equation (2.24b). This yields two integration functions: the first of them, of order r^0 , must vanish as a consequence of Sommerfeld's condition, i.e. only outwards flowing radiation solutions are allowed.
3. Solve Equation (2.24c) with the results gathered thus far to obtain V up to the function $M(u, x^A)$ mentioned above. This effectively solves the system for the hypersurface $u = u_0$.
4. Evolve the solution using Equation (2.26), which yields h_{AB} up to the c_{AB} integration function.

For General Relativity, the presented algorithm yields

$$\beta = -\frac{1}{32} \frac{c^{AB} c_{AB}}{r^2} + \dots, \quad (2.30a)$$

$$U^A = -\frac{\partial_B c^{AB}}{2r^2} + \frac{1}{r^3} \left(2L^A + \frac{1}{3} c^{AE} \partial^F c_{EF} \right) + \dots, \quad (2.30b)$$

$$V = r - 2M + \dots \quad (2.30c)$$

$$h_{AB} = q_{AB} + \frac{c_{AB}}{r} + \frac{d_{AB}}{r^2} + \dots \quad (2.30d)$$

When we later restrict our focus to the axially symmetric Bondi metric, we shall show that the function M is closely related to the mass of the system. In particular, the E_u^r supplementary condition yields

$$\partial_u M = -\frac{1}{8} N_{AB} N^{AB} + \frac{1}{4} \bar{\partial}_A \bar{\partial}_B N^{AB}, \quad (2.31)$$

where the so-called news tensor is defined as

$$N_{AB} = \partial_u c_{AB}. \quad (2.32)$$

Equation (2.31) is known as the Bondi mass loss formula, and it is the main result we aim to replicate for alternative theories of gravity using our program. It is related to the time-dependent Bondi mass $m(u)$ by the integral over the sphere

$$m(u) = \frac{1}{4\pi} \oint M(u, \theta, \phi) \sin \theta \, d\theta \, d\phi, \quad (2.33)$$

which explains the name of news tensor through its connection to the flow of energy out of the system. Or, as Bondi formulated in his 1962 paper [10]:

The mass of a system is constant if and only if there is no news. If there is news, the mass decreases monotonically as long as the news continues.

The second supplementary condition E_A^r yields

$$\begin{aligned} -3 \partial_u L_A = & \bar{\partial}_A M - \frac{1}{4} \bar{\partial}^E (\bar{\partial}_E \bar{\partial}^F c_{AF} - \bar{\partial}_A \bar{\partial}^F c_{EF}) + \frac{1}{8} \bar{\partial}_A (c_{EF} N^{EF}) \\ & - \bar{\partial}_C (c^{CF} N_{FA}) + \frac{1}{2} c^{EF} (\bar{\partial}_A N_{EF}), \end{aligned} \quad (2.34)$$

where L_A is called angular momentum aspect. The connection of L_A with angular momentum shall also become clear later when we consider the axial symmetric case of the Bondi-Sachs metric.

Though the inclusion of additional fields certainly changes some aspects of the above derivation, in the following chapters we will base much of the discussion on the results presented in these sections.

2.3 The Case of Electrodynamics

Before considering the relevant changes when working in the case of axial symmetry, in this section we make a brief excursion into electrodynamics where the ideas of the formalism are also applicable. With this, we use the example of a linear theory to motivate the notion that the change in mass of a system, in the GR case, is a consequence of the non-linear nature of Einstein's equations. However, it must be noted that this is not strictly a proof, as charge conservation in the case of electrodynamics is not exclusively related to the linear nature of Maxwell's equations, but also the gauge freedom of the vector potential [21].

This section closely follows the Section 2.1 of [25], and has been included in the interest of keeping this thesis self-contained for readers not already familiar with the topic.

In flat spacetime electrodynamics, we are working with a Minkowski metric. Similar to GR, we interest in studying radiation propagating along the lightcone; thus, we consider the flat space version of the Bondi-Sachs metric,

$$\eta_{ab} dx^a dx^b = -du^2 - 2dr du + r^2 q_{AB} dx^A dx^B. \quad (2.35)$$

The electromagnetic field can be derived from the vector potential A_a , and is described by the tensor

$$F_{ab} = \nabla_a A_b - \nabla_b A_a, \quad (2.36)$$

The vector potential has the gauge freedom

$$A_a \rightarrow A_a + \nabla_a \chi. \quad (2.37)$$

The system is assumed to be isolated in a way such that all sources are enclosed in a timelike worldtube Γ of radius $r = R$. We introduce the null gauge $A_r = 0$, analogous to $g_{rr} = g_{rA} = 0$ in GR, by the gauge transformation

$$\chi(u, r, x^A) = - \int_R^r A_r dr'. \quad (2.38)$$

Through an additional gauge transformation with $\chi(u, x^A)$ we set either

$$A_u|_\Gamma = A_u(u, R, x^A) = 0 \quad \text{or} \quad \lim_{r \rightarrow \infty} A_u(u, r, x^A) = 0, \quad (2.39)$$

where limits $r \rightarrow \infty$ are taken with $u, x^A = \text{const.}$. There is the remaining gauge freedom $A_B \rightarrow A_B + \nabla_B \chi(x^C)$.

Just as Equation (2.13) encodes Einstein's equations, the vacuum Maxwell equations can be encoded as

$$M^b = \nabla_a F^{ab} = 0, \quad (2.40)$$

which implies the identity analogous to Equation (2.16)

$$0 = \nabla_b M^b = \partial_u M^u + \frac{1}{r^2} \partial_r (r^2 M^r) + \frac{1}{\sqrt{\det q}} \partial_C (\sqrt{\det q} M^C). \quad (2.41)$$

We designate as the main equations $M^u = 0$ and $M^A = 0$, and denote $M^r = 0$ as supplementary condition. Similar to the GR case, if the main equations satisfied, then (2.41) implies

$$0 = \partial_r (r^2 M^r); \quad (2.42)$$

hence, the supplementary condition is satisfied everywhere, provided that it is satisfied for some value of r , which in our construction is either at the worldtube Γ or for $r \rightarrow \infty$.

The main equations in electrodynamics can also be separated into the hypersurface equation given by

$$M^u = 0 : \quad \partial_r r^2 \partial_r A_u = \partial_r (\eth_B A^B), \quad (2.43)$$

and the evolution equation

$$\begin{aligned} M^A = 0 : \quad \partial_r (\partial_u A_B) &= \frac{1}{2} \partial_r^2 A_B \\ &\quad - \frac{r^2}{2} \eth^C (\eth_B A_C - \eth_C A_B) + \frac{1}{2} \partial_r (\eth_B A_u). \end{aligned} \quad (2.44)$$

The supplementary condition can be expanded into

$$M^r = 0 : \quad \partial_u (r^2 \partial_r A_u) = \partial^B (\partial_r A_B - \partial_u A_B + \partial_B A_u) \quad (2.45)$$

Now, the hypersurface equation (2.43) can be formally integrated as

$$\partial_r A_u = \frac{Q(u, x^A) + \partial_B A^B}{r^2} + \dots, \quad (2.46)$$

where $Q(u, x^A)$ is a integration function. For our choice of gauge, we have $E_r = F_{ru} = \partial_r A_u$; hence, the total charge enclosed in the system is given by the integral

$$q(u) = \lim_{r \rightarrow \infty} \frac{1}{4\pi} \oint E_r r^2 \sin \theta d\theta d\phi = \frac{1}{4\pi} \oint Q(u, x^A) \sin \theta d\theta d\phi, \quad (2.47)$$

where the $\partial_B A^B$ term can be eliminated using the divergence theorem. The definition of $q(u)$ motivates the name charge aspect for $Q(u, x^A)$.

The integral over the sphere of the supplementary condition (2.45) yields

$$\begin{aligned} & \oint \partial_u (Q(u, x^A)) r^2 \sin \theta d\theta d\phi \\ &= \oint \partial^B (\partial_r A_B - \partial_u A_B + \partial_B A_u) d^2 x = 0, \end{aligned} \quad (2.48)$$

where we again use the divergence theorem to eliminate the right side. This gives the charge conservation law

$$\frac{dq(u)}{du} = 0. \quad (2.49)$$

Thus, contrary to the mass of a system in GR, in electrodynamics though the system may loose energy through the emission of electromagnetic radiation, the charge contained inside the system, i.e. monopole moment, is conserved.

2.4 Axial Symmetry

In its current state, Maxima does not provide any mature packages for tensor manipulation and our developed code is bound to working explicitly in terms of a specific choice of coordinates. Thus, we restrict our discussion to an axisymmetric system, in particular, one described by the original metric provided by Bondi, where our results can be easily checked against the originals. Nevertheless, the current code is not dependent on this particular choice and is also capable of computing the same results for other systems of coordinates. For example, the metric introduced in [39] with a spherical part given by

$$\begin{aligned} h_{AB} dx^A dx^B &= (e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta d\phi^2) \cosh(2\delta) \\ &\quad + 2 \sin \theta \sinh(2\delta) d\theta d\phi \end{aligned} \quad (2.50)$$

However, as mentioned above, we use for the 2-sphere the metric introduced by Bondi

$$h = e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta d\phi^2. \quad (2.51)$$

Thus, the Bondi-Sachs metric (2.4) becomes

$$g_{ab}^{(B)} dx^a dx^b = (U^2 r^2 e^{2\gamma} - V r^{-1} e^{2\beta}) du^2 - 2e^{2\beta} du dr - 2U r^2 e^{2\gamma} du d\theta + r^2 (e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta d\phi^2), \quad (2.52)$$

known also as the Bondi metric first given in [9].

Similar to the more general case, the equations of motion can be solved, order-by-order, using the same type of series expansion in r^{-1} as outlined above in Equations (2.24a)–(2.24c) and (2.26). The major difference is that the evolution equation for h_{AB} is simplified to a evolution for a scalar instead of tensor. In total, this yields

$$\gamma = c(u, \theta) r^{-1} + \left[C(u, \theta) - \frac{1}{6} c^3 \right] + \dots \quad (2.53a)$$

$$\beta = -\frac{1}{4} c^2 r^{-2} - \frac{3}{4} c \left(C - \frac{1}{6} c^3 \right) r^{-4} + \dots \quad (2.53b)$$

$$U = -(c_\theta + 2c \cot \theta) r^{-2} + [2N(u, \theta) + 3cc_\theta + 4c^2 \cot \theta] r^{-3} + \frac{1}{2} (3C_\theta + 6C \cot \theta - 6nN - 8c^2 c_\theta - 8c^3 \cot \theta) r^{-4} + \dots \quad (2.53c)$$

$$V = r - 2M(u, \theta) - \left[N_\theta + N \cot \theta - c_\theta^2 - 4cc_\theta \cot \theta - \frac{1}{2} c^2 (1 + 8 \cot^2 \theta) \right] r^{-1} - \frac{1}{2} [C_{\theta\theta} + 3C_\theta \cot \theta - 2C + 6N(c_\theta + 2c \cot \theta) + 8c(c_\theta^2 + 3cc_\theta + 2c^2 \cot^2 \theta)] r^{-2} + \dots \quad (2.53d)$$

The r^{-2} coefficient of the evolution equation yields the loss formula for what we shall show to be the quadrupole moment

$$4C_u = 2c^2 c_u + 2cM + N \cot \theta - N_\theta \quad (2.54)$$

As expected, the supplementary conditions yield the mass-loss and momentum-loss formulae

$$M_u = -c_u^2 + \frac{1}{2} (c_{\theta\theta} + 3c_\theta \cot \theta - 2c)_u \quad (2.55)$$

$$-3N_u = M_\theta + 3cc_{u\theta} + 4cc_u \cot \theta + c_u c_\theta \quad (2.56)$$

For later reference, we also present the wave-operator in Bondi coordinates, given by

$$\begin{aligned} \square \Phi &= \frac{e^{-2\gamma}}{r^2} (\Phi_\theta \cot \theta - 2\Phi_\theta \gamma_\theta + 2\Phi_\theta \beta_\theta + \Phi_{\theta\theta}) + \frac{1}{r^2} V \Phi_r e^{-2\beta} \\ &\quad + \frac{e^{-2\beta}}{r} (-2U \Phi_\theta - 2\Phi_u + V \Phi_{rr} + V_r \Phi_r) \\ &\quad - e^{-2\beta} (U \Phi_r \cot \theta + 2U \Phi_{r\theta} + 2\Phi_{ru} + U_\theta \Phi_r) - U_r \Phi_\theta \end{aligned} \quad (2.57)$$

Furthermore, for an arbitrary function f on the 2-sphere, the Laplacian is given by:

$$\mathbb{D}^2 f = f_{\theta\theta} + f_\theta \cot \theta + f_{\phi\phi} \csc^2 \theta. \quad (2.58)$$

2.5 Meaning of the Metric Aspects

In the previous sections, we have claimed that the metric functions M and N are somehow related to the mass and angular momentum of the isolated system. In this section, we make this connection—in addition to the meaning of the function C —clearer.

We first imagine a period in the system’s evolution where the system is static, and transitions at some $u = u_0$ to a dynamical state in which gravitational waves are emitted. In our case of axial symmetry, we can describe the period $u < u_0$ through the Weyl metric

$$g_{ab}^{(W)} dx^a dx^b = -e^{2\psi} dt^2 + e^{2\sigma-2\psi} (dR^2 + R^2 d\Theta^2) + e^{-2\psi} R^2 \sin^2 \Theta d\phi^2, \quad (2.59)$$

which is at this point not yet expressed in retarded time form. The system’s later evolution for $u > u_0$ can be described by the axi-symmetric Bondi metric (2.52). Our task is now to show that one can find a suitable coordinate transformation that converts (2.59) into a case of the Bondi metric, such that the evolution of the system can be connected at the point $u = u_0$ and the coefficients can be meaningfully translated.

The result presented here follows closely Appendix 4 of Bondi’s paper [10]; the notable difference is that we provide a more detailed explanation of the derivation’s steps. The reason to include this section is twofold. First, in our opinion, the result is vital to really understand the meaning of the mass-loss formula, and the reader is not assumed to be already familiar with the formalism. Also, the original derivation is—again, in our opinion—confusing enough that we consider it sensible to help the reader understand it quicker.

Also, note that the following results were not blindly copied from the original paper: the code accompanying this thesis includes a program with a step by step derivation.

We are only interested in the fact that the metric can be translated; not necessarily in how all components actually look like in their Bondi form. Hence, as in the original publication, to simplify computations, we split the procedure in two different coordinate transformations. We introduce new coordinates defined by the transformations

$$t = u + f(R, \theta) \quad \text{and} \quad \Theta = g(R, \theta). \quad (2.60)$$

According to the definition of the Bondi metric, we require the metric to satisfy $g_{RR} = 0$ and $g_{R\theta} = 0$. This yields the equations

$$e^{2\psi} f_R^2 = e^{2\sigma-2\psi} [1 + R^2 g_R^2] \quad \text{and} \quad e^{2\psi} f_R f_\theta = e^{2\sigma-2\psi} R^2 g_R g_\theta. \quad (2.61)$$

We eliminate f by combining the equations and building $(f_R^2)_\theta = (f_\theta^2)_R$. This results in

$$R^2 g_R (e^{2\sigma-4\psi})_\theta = g_\theta \left[e^{2\sigma-4\psi} \frac{R^4 g_R^2}{1 + R^2 g_R^2} \right]. \quad (2.62)$$

In the literature, the solutions of the Weyl metric are known to be expressible as a multipole expansion in powers of $1/R$, where the coefficients have known interpretations as the mass, dipole moment, etc. [10] A brief derivation of this fact is also given in Appendix A. From this, we may express the exponential factor as

$$e^{2\sigma-4\psi} = 1 + \frac{4m}{R} + \frac{p(\Theta)}{R^2} + \frac{q(\Theta)}{R^3} + \dots, \quad (2.63)$$

for some function p and q of Θ . We take the case $c = 0$, where c is the same function introduced in the previous section. This corresponds to

$$\lim_{R \rightarrow \infty} g_\theta = 1. \quad (2.64)$$

Thus, expressing g as a power series in $1/R$, (2.62) yields

$$g = \theta + \frac{p'}{4R^2} + \frac{q' - 6mp'}{23R^3} + \dots, \quad (2.65)$$

from which f can also be solved for

$$f = R + 2m \log R - \frac{\frac{1}{2}p - 2m^2}{R} - \frac{\frac{1}{2}q - mp + 4m^3}{2R^2} + \dots \quad (2.66)$$

As we see from this last result, the fact that we are dealing with radial derivatives of f , rather than f itself, means we also have to account for possible logarithmic terms. Also, from (2.60), it is clear that f has to have terms at most of order R , for the resulting u coordinate to asymptotically go towards the retarded time $u = t - R$.

We concatenate a second coordinate transformation by introducing a r coordinate satisfying the areal coordinate requirement

$$r^4 \sin^2 \theta = g_{\theta\theta} g_{\phi\phi} = e^{-2\psi} R^2 \sin^2 \Theta [e^{2\sigma-2\psi} R^2 g_\theta^2 - e^{2\psi} f_\theta^2]. \quad (2.67)$$

Assuming r to be expressible as a power series in $1/R$, we can solve Equation (2.67) for r using also the Taylor series and sum of arguments identity for the sine function

$$\begin{aligned} \sin \Theta = \sin \theta \cos \left(\frac{p'}{4R^2} + \frac{q' - 6mp'}{23R^3} + \dots \right) \\ + \cos \theta \sin \left(\frac{p'}{4R^2} + \frac{q' - 6mp'}{23R^3} + \dots \right). \end{aligned} \quad (2.68)$$

Hence, we have

$$\begin{aligned} r = R + m + \frac{p_{\theta\theta} + p_\theta \cot \theta + 2p - 12m^2}{8R} \\ + \frac{q_\theta \cot \theta - 3mp_\theta \cot \theta + q_{\theta\theta} + 6q - 3mp_{\theta\theta} - 18mp + 84m^3}{24R^2} + \dots \end{aligned} \quad (2.69)$$

Because both r and R describe some radius, we further assume that asymptotically

$$\lim_{r \rightarrow \infty} \frac{R(r)}{r} = \lim_{R \rightarrow \infty} \frac{r(R)}{R} = 1, \quad (2.70)$$

such that r and R are related by transformations of the form

$$r = a_{-1}R + a_0 + \frac{a_1}{R} + \frac{a_2}{R^2} + \dots, \quad (2.71)$$

$$R = b_{-1}r + b_0 + \frac{b_1}{r} + \frac{b_2}{r^2} + \dots, \quad (2.72)$$

and we can solve for $1/r$ by using

$$\frac{1}{r} = \frac{1}{a_{-1}R + a_0 + a_1/R + \dots} = \frac{x_1}{R} + \frac{x_2}{R^2} + \dots \quad (2.73)$$

By definition of the inverse, we must have

$$r \cdot \frac{1}{r} = \left(a_{-1}R + a_0 + \frac{a_1}{R} + \dots \right) \left(\frac{x_1}{R} + \frac{x_2}{R^2} + \dots \right) = 1. \quad (2.74)$$

Solving term by term yields

$$\frac{1}{r} = \frac{1}{R} - \frac{m}{R^2} - \frac{p_\theta \cot \theta + p_{\theta\theta} + 2p - 20m^2}{8R^3} + \dots \quad (2.75)$$

Plugging everything into the transformation, we have the following equation

$$\begin{aligned} R = b_{-1} \left(a_{-1}R + a_0 + \frac{a_1}{R} + \frac{a_2}{R^2} + \dots \right) \\ + b_0 + b_1 \left(\frac{x_1}{R} + \frac{x_2}{R^2} + \dots \right) + b_2 \left(\frac{x_1}{R} + \frac{x_2}{R^2} + \dots \right)^2 + \dots \end{aligned} \quad (2.76)$$

Comparing coefficients, we can solve for the b_n . Finally, R is given by

$$\begin{aligned} R = r - m - \frac{1}{2} \left[\frac{1}{4} p' \cot \theta + \frac{1}{2} p - 3m^2 + \frac{1}{4} p'' \right] r^{-1} \\ - \frac{1}{2} \left[\frac{1}{12} q'' + \frac{1}{12} q' \cot \theta + \frac{1}{2} q - mp + 4m^3 \right] r^{-2} + \dots \end{aligned} \quad (2.77)$$

Using the series expansions provided in Appendix A, one can substitute for ψ and σ in (2.63) and find the explicit values for p and q :

$$p = 4D \cos \Theta + m^2 (7 + \cos^2 \Theta), \quad (2.78)$$

$$q = 2Q (3 \cos^2 \Theta - 1) + 4mD (3 + \cos^2 \Theta) + 6m^3 (1 + \cos^2 \Theta). \quad (2.79)$$

Hence,

$$R = r - m - \frac{1}{2} m^2 (1 - \cos^2 \theta) r^{-1} - \frac{1}{2} m \sin^2 \theta [2D \cos \theta + m^2] r^{-2} + \dots \quad (2.80)$$

Finally,

$$\frac{V}{r} e^{2\psi} f_R R_r = e^{2\psi} + e^{2\psi} \frac{(f_R R_\theta + f_\theta)^2}{R^2 e^{2\sigma - 4\psi} g_\theta^2 - f_\theta^2}. \quad (2.81)$$

Thus,

$$\frac{V}{r} = 1 - \frac{2m}{r} - \frac{2D \cos \theta}{r^2} - \frac{Q(3 \cos^2 \theta - 1)}{r^3} + \dots \quad (2.82)$$

Comparing with Equation (2.53d), we have

$$M = m, \quad N = D \sin \theta, \quad C = \frac{1}{2} Q \sin^2 \theta. \quad (2.83)$$

Values for $c \neq 0$ can be generated by a transformation of the BMS group [10].

From the results of this section, it should now be clear how the aspect functions are related, at least in the axial symmetric case, to different physical observables of the system. Together with their conservation equations, we also have a description of how the mass, and other parameters, of the isolated gravitational system change over time due to the emission of gravitational waves.

Chapter 3

Scalar-tensor Theories of Gravity

In this Chapter, we briefly present the background and motivations behind scalar-tensor theories of gravity, their connection with earlier attempts at unification theories. In Section 3.1, we introduce Horndeski action [19], which encodes the most general theory with a scalar field coupled to the metric in four-dimensional spacetime that yields second order equations of motion, but continue the discussion with a subset of the theory given by the simpler action, known as Bergmann-Wagoner [4, 31]. The concept of a conformal transformation of the metric is explained in Section 3.2. This is specially relevant for later chapters of this thesis. Moreover, we briefly comment on the existing discussion around the physical nature of the two main conformal frames existing in the literature. Section 3.3 closes this chapter by arguing for the relevance in applying the Bondi-Sachs formalism to scalar-tensor theories of gravity.

3.1 Introduction

Given the success of the theory of General Relativity, there is a clear motivation to describe gravity—at least in an effective sense—through a field theory based on a metric, i.e. non-degenerate bilinear form at each point of a 4 dimensional manifold. One of the simplest ways to extend such a theory, other than maybe including a cosmological constant to the field equations, is to introduce a scalar field to the theory. Of course, our goal is for this to result in a modified theory; not just the postulation of an additional “matter” field. Therefore, we do not consider minimally coupled theories, like those stemming from the action

$$S[g, \Phi] = \frac{1}{16\pi} \int \left[R - \frac{1}{2} g^{ab} \partial_a \Phi \partial_b \Phi - \Pi(\Phi) \right] \sqrt{|\det g|} \, d^4x. \quad (3.1)$$

The family of theories we seek must possess, in general, a scalar field non-minimally coupled to gravity. Historically, one of the earlier examples for such theories was proposed by Jordan [22]. In his theory, a four-dimensional curved manifold would be embedded in a five-dimensional flat spacetime [15]. In this construction, it was possible to describe a spacetime dependent, and in particular, P. A. M. Dirac’s argument that gravitational “constant” should be time-

dependent [14, 15]. Jordan also discussed the possible connection of his work with the Kaluza-Klein theory, a unification of gravity with electrodynamics in also a five-dimensional manifold and precursor to modern string-theory.

In its time, Jordan's theory was criticized by the further assumptions required for the introduction of matter and the non-conservation of the energy-momentum tensor. Also, it in principle allowed for the violation of the weak-equivalence principle, which for us here states the local equality of all gravitational accelerations and is supported by Eötvös' experiments [11]. In more recent literature, the simplest example of a scalar-tensor theory is the closely related Brans-Dicke theory, which does respect the WEP albeit not the strong equivalence principle—that the laws of physics, as observed locally in a freely falling laboratory, are independent of the location in time or space of said laboratory [11].

For the purposes of our software, we are potentially interested in covering all imaginable theories of gravity. Thus, in this work, it is sensible to at the very least mention the most general scalar-tensor theory in four-dimensional space-time that leads to second order equations of motion. This last restriction is by no means arbitrary; third order equations of motion originate from degenerate Lagrangians and fourth order ones from, possible non-degenerate, Lagrangians with second-order derivatives in time. In both cases the resulting Hamiltonian is unbounded from below (see Ostrogradski's Theorem and [43, 27]), and may be undesired for a physical theory.

The theory describing what we are interested in is called Horndeski theory, and is given by the action [19, 18]

$$S[g, \Phi] = \sum_{i=2}^5 \int \mathcal{L}_i[g_{ab}, \Phi] \sqrt{|\det g|} d^4x + S_m[g, \chi_m], \quad (3.2)$$

where

$$\mathcal{L}_2 = K(\Phi, X), \quad (3.3)$$

$$\mathcal{L}_3 = -G_3(\Phi, X) \square \Phi, \quad (3.4)$$

$$\mathcal{L}_4 = G_4(\Phi, X) R + G_{4X}(\Phi, X) [(\square \Phi)^2 - (\nabla_a \nabla_b \Phi)^2], \quad (3.5)$$

$$\begin{aligned} \mathcal{L}_5 = & G_5(\Phi, X) G_{ab} \nabla^a \nabla^b \Phi \\ & - \frac{1}{6} G_{5X}(\Phi, X) [(\square \Phi)^3 - 3(\square \Phi)(\nabla_a \nabla_b \Phi)^2 + 2(\nabla_a \nabla_b \Phi)^3]. \end{aligned} \quad (3.6)$$

We also make use of the notation

$$\begin{aligned} \square &= g^{ab} \nabla_a \nabla_b, \\ X &= -\frac{1}{2} \nabla^a \Phi \nabla_a \Phi, \\ (\nabla_a \nabla_b \Phi)^2 &= \nabla_a \nabla_b \Phi \nabla^a \nabla^b \Phi, \\ (\nabla_a \nabla_b \Phi)^3 &= \nabla_a \nabla_b \Phi \nabla^b \nabla^c \Phi \nabla_c \nabla^a \Phi. \end{aligned}$$

The functions K , G_3 , G_4 and G_5 are arbitrary and define the theory. We remark that in this context, the second subindex represents a partial derivative with respect to the kinetic term X .

However, for this thesis, it suffices if we first consider the smaller family of theories, sometimes called Bergmann-Wagoner [4, 31], and given by the action

$$S[g, \Phi] = \frac{1}{16\pi} \int [\Phi R - B(\Phi) g^{ab} \partial_a \Phi \partial_b \Phi - \Pi(\Phi)] \sqrt{|\det g|} d^4x, \quad (3.7)$$

where the function B , coupled to the kinetic term, and the potential Π , are arbitrary functions of Φ .

This covers the well-known Brans-Dicke theory—for which an analysis in Bondi-Sachs form has already been made by other authors [36, 20]—as well as other theories of interest, some of which are also covered in this thesis.

3.2 Conformal Transformation of Scalar-Tensor Theories

A scalar-tensor theory of gravity is said to be expressed in the Jordan frame if the scalar field is non-minimally coupled by means of a product to the scalar curvature in the action, like in (3.7). For our discussion it is useful to isolate the scalar curvature through a conformal transformation, such that the resulting scalar field is only minimally coupled to the transformed metric. This is called the Einstein frame. To keep our discussion general, we use the action

$$S[g, \Phi] = \frac{1}{16\pi} \int [A(\Phi)R - B(\Phi) g^{ab} \partial_a \Phi \partial_b \Phi - \Pi(\Phi)] \sqrt{|\det g|} d^4x, \quad (3.8)$$

with A , B and Π arbitrary functions of Φ . We introduce a new metric $\tilde{g}_{ab}$ defined by the transformation:

$$\tilde{g}_{ab} = \Omega(x)^2 g_{ab}.$$

We derive its metric compatible connection from the Christoffel-like symbols defined from

$$\tilde{\nabla}_a \omega_b = \nabla_a \omega_b - C_{ab}^c \omega_c$$

as

$$C_{ab}^c = 2\delta_{(b}^c \partial_{a)} f - g_{ab} \partial^c f,$$

where

$$f = \ln \Omega.$$

Thus, the scalar curvature related to $\tilde{g}_{ab}$ is given by [41]

$$\tilde{R} = \Omega^{-2} [R - 6\Box f - 6g^{ab} \partial_a f \partial_b f]$$

which can be inverted into

$$R = \Omega^2 [\tilde{R} + 6\tilde{\Box} f - 6\tilde{g}^{ab} \partial_a f \partial_b f]. \quad (3.9)$$

We now let $A\Omega^{-2} = 1$, substitute Equation (3.9) into Equation (3.8) and drop the $\tilde{\Box} f$ term, as it be rewritten as divergence [15]

$$\tilde{\Box} f = \frac{1}{\sqrt{|\det \tilde{g}|}} \frac{\partial}{\partial x^a} \left(\sqrt{|\det \tilde{g}|} \tilde{g}^{ab} \partial_b f \right). \quad (3.10)$$

By noting that

$$\partial_a f = \frac{1}{2} \frac{d \ln A}{d\Phi} \partial_a \Phi,$$

we have now the action

$$\frac{1}{16\pi} \int \left[\tilde{R} - \left(\frac{3}{2} \left[\frac{d \ln A}{d\Phi} \right]^2 + \frac{B(\Phi)}{A(\Phi)} \right) \tilde{g}^{ab} \partial_a \Phi \partial_b \Phi - \frac{\Pi(\Phi)}{A(\Phi)} \right] \sqrt{|\det \tilde{g}|} d^4 x.$$

Hence, the action (3.8) can be converted into the so-called Einstein frame action

$$S_E[\tilde{g}, \varphi] = \frac{1}{16\pi} \int \left[\tilde{R} - \frac{1}{2} \tilde{g}^{ab} \partial_a \varphi \partial_b \varphi - \tilde{\Pi}(\varphi) \right] \sqrt{|\det \tilde{g}|} d^4 x \quad (3.11)$$

by means of the transformation

$$\left. \begin{aligned} \tilde{g}_{ab} &= A(\Phi) g_{ab}, \\ \frac{1}{2} \left(\frac{d\varphi}{d\Phi} \right)^2 &= \frac{3}{2} \left(\frac{d \ln A}{d\Phi} \right)^2 + \frac{B(\Phi)}{A(\Phi)}, \\ \tilde{\Pi}(\varphi) &= \frac{\Pi(\Phi)}{A(\Phi)^2}. \end{aligned} \right\} \quad (3.12)$$

Though the form of the Einstein frame is reminiscent of the Einstein-Hilbert action with a minimally coupled scalar field, there is the important distinction: in this conformal frame, test particles no longer follow geodesics, but are instead “accelerated”. In its simplest case, this is directly seen by minimally coupling an arbitrary set of matter fields to the Jordan frame action

$$S[g, \Phi, \chi_m] = S_J[g, \Phi] + S_m[g, \chi_m]$$

which is then mapped into

$$S[\tilde{g}, \varphi, \chi_m] = S_E[\tilde{g}, \varphi] + S_m[A^{-1} \tilde{g}, \chi_m] \quad (3.13)$$

It is clear that a conformal transformation should not be confused by a change in coordinates. Also, we make the following simple, but important observation due to the scaling property of a conformal transformation: *the potential in the action of a theory cannot be eliminated through a change of conformal frame.*

The last statement plays an important role in later chapters. Its significance is twofold. Firstly, it means that all actions with a potential term in Jordan frame are mapped to a corresponding action in the Einstein frame that necessarily also has a potential; hence, by contraposition, if one would prove impossible to find a Bondi mass-loss formula for a general potential in Einstein frame, then every theory described with a potential in the Jordan frame would also have no Bondi mass-loss formula. Secondly, every theory missing a potential term in the Jordan frame is transformed into the same form in the Einstein frame; thus, the result of carrying out a series expansion in the Einstein frame can be used for all theories without potential in the Jordan frame by choosing the appropriate conformal transformation.

3.3 Why a Mass-loss Formulae for Scalar-Tensor Theories?

We close this chapter by reflecting on an important question about our motivation:

Why should one look into mass-loss formulae for scalar-tensor theories of gravity?

It is known from PPN expansions that alternative theories of gravity sometimes introduce different polarization modes for gravitational waves (see breathing modes in [42]). Moreover, as a new degree of freedom is introduced into the theory, we expect it to influence how energy is radiated away from an isolated gravitational system.

Clearly, there is reason to believe that a change in the mass-loss formula is to be expected in scalar-tensor theories. This has in fact been proven for at least the case of the Brans-Dicke theory, as we shall see in the following chapter.

The changes in mass-loss formulae could eventually be used to make concrete and measurable predictions on the energy irradiated by a system, which can be used as a method to validate or discard alternative theories of gravity.

Finally, whether modification of General Relativity can alter the structure of the BMS group is still an open question. Though, for Brans-Dicke this is not the case [36], a general result is still missing, and would be of great theoretical relevance. The mass-loss formula together with the structure of the BMS group determine gravitational-wave memory effects, which again can be used to make measurable predictions from a theory.

Chapter 4

Brans-Dicke Theory

The Bondi-Sachs formalism in the context of Brans-Dicke theory—one of the simpler examples of a scalar-tensor theory—has been the topic of at least two separate recent publications [20, 36]. It provides an approachable case study of the differences from the usual GR treatment. Though topics like the exact requirements for the asymptotic behavior of the theory remain debated, it is a relevant result that we shall use to base our search for mass-loss formulae in other, possibly more complex, theories of gravity.

This chapter is intended to provide a summary of known results regarding the Bondi-Sachs formalism in Brans-Dicke theory. Similar to [36], this chapter starts in Section 4.1 by deriving the asymptotic boundary conditions in the Einstein frame—where the field equations have the same form as the Einstein-Klein-Gordon equations for a massless scalar field—and translating them into the Jordan frame, where a mass-loss formula is of more experimental use. Section 4.2 continues by using the information gathered in the previous section and solving the field equations once again in the Jordan frame. Finally, in Section 4.3, the loss formulae for Brans-Dicke theory are presented and compared with the axially symmetric results that our program yields. This provides a final comparison with existing literature, before our program can be used to derive novel results.

4.1 Brans-Dicke Theory in the Einstein Frame

The action of Brans-Dicke theory in the Einstein frame is given by [36]

$$S_E^{\text{BD}}[\tilde{g}, \varphi] = \int \left(\frac{\tilde{R}}{16\pi} - \frac{1}{2} \tilde{g}^{ab} \partial_a \varphi \partial_b \varphi \right) \sqrt{|\det \tilde{g}|} \, d^4x. \quad (4.1)$$

We use tilde to denote the Einstein frame version of the different objects used. Though a conformal transformation changes only the metric, we denote the set of coordinates by $\tilde{x}^a = (\tilde{u}, \tilde{r}, x^A)$; as we shall later see in this chapter, an additional change of coordinates is needed to ensure the Bondi gauge when transforming into the Jordan frame.

The equations of motion, with the addition of a equation for the evolution

of the scalar field, take the same form of Equation (2.13):

$$0 = \tilde{E}_{ab} = \tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} - 8\pi\tilde{T}_{ab}^{\text{eff}} \quad (4.2)$$

$$0 = \tilde{\square}\varphi, \quad (4.3)$$

where the wave operator and stress-energy tensors are given respectively by

$$\tilde{\square} = \tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b \quad \text{and} \quad \tilde{T}_{ab}^{\text{eff}} = \partial_a\varphi\partial_b\varphi - \frac{1}{2}\tilde{g}_{ab}\tilde{g}^{cd}\partial_c\varphi\partial_d\varphi. \quad (4.4)$$

The construction of the metric follows the same steps as in GR. We work in the Bondi gauge with

$$\tilde{g}_{\tilde{r}A} = \tilde{g}_{\tilde{r}\tilde{r}} = 0 \quad \text{and} \quad \det g = \tilde{r}^4 \det q; \quad (4.5)$$

and introduce the Bondi-Sachs metric in its usual form

$$\begin{aligned} \tilde{g}_{ab} d\tilde{x}^a d\tilde{x}^b = & \frac{\tilde{V}}{\tilde{r}} e^{2\tilde{\beta}} d\tilde{u}^2 - 2e^{2\tilde{\beta}} d\tilde{u} d\tilde{r} \\ & + \tilde{r}^2 \tilde{h}_{AB} \left(d\tilde{x}^A - \tilde{U}^A d\tilde{u} \right) \left(d\tilde{x}^B - \tilde{U}^B d\tilde{u} \right). \end{aligned} \quad (4.6)$$

The algorithm we introduced in Chapter 2 to solve the equations of motion has to be modified to account for the presence of the scalar field as follows: let us now assume for $u = u_0$ that not only the value of h_{AB} is known, but also that of φ . As in GR, Equations (2.24a) to (2.24c) are used to compute β , U and V on the hypersurface $u = u_0$; however, together with Equation (2.26), we also have to use Equation (4.3) to evolve the system. The expanded form of the equations of motion was already given in Chapter 2, here we provide only the expansion for the evolution equation of the scalar field, given by:

$$\begin{aligned} & 2\partial_{\tilde{u}}\partial_{\tilde{r}}\varphi + \tilde{D}_A(\tilde{U}^A\partial_{\tilde{r}}\varphi) + \partial_{\tilde{r}}\tilde{U}^A\tilde{D}_A\varphi \\ & - \frac{1}{\tilde{r}} \left(-2\tilde{U}^A\tilde{D}_A\varphi - 2\partial_{\tilde{u}}\varphi + \partial_{\tilde{r}}\tilde{V}\partial_{\tilde{r}}\varphi + \tilde{V}\partial_{\tilde{r}}\partial_{\tilde{r}}\varphi \right) \\ & - \frac{1}{\tilde{r}^2} \left(e^{2\tilde{\beta}}\tilde{h}^{AB} \left[2\tilde{D}_A\beta\tilde{D}_B\varphi + \tilde{D}_B\tilde{D}_A\varphi \right] + \tilde{V}\partial_{\tilde{r}}\varphi \right) = 0. \end{aligned} \quad (4.7)$$

In addition to the known series expansions for the parts of the metric given by Equations (2.29a)–(2.29d), we introduce an expansion of φ in powers of r^{-1} of the form

$$\varphi(\tilde{u}, \tilde{r}, \tilde{x}^A) = \varphi_0 + \frac{\varphi_1(\tilde{u}, \tilde{x}^A)}{\tilde{r}} + \dots \quad (4.8)$$

The coefficient φ_0 can be shown to be constant; for an expansion of β , U , V and h_{AB} consistent with the main equations, the coefficient r^{-1} of Equation (4.7) implies $\partial_{\tilde{u}}\varphi_0 = 0$. Similarly, the coefficient r^{-2} implies $\partial_{\tilde{r}}\varphi_0 = 0$; in particular, in the axial symmetric case we have $(\varphi_0)_{\theta\theta} + (\varphi_0)_{\theta} \cot \theta = 0$ where the only solution that remains bounded inside the sphere is $\varphi_0 = \text{const.}$

Like in GR, we assume the functions $\tilde{\beta}$, $\tilde{U}^A$, $\tilde{V}$ and $\tilde{h}_{AB}$ have the following limits for $\tilde{r} \rightarrow \infty$:

$$\lim_{\tilde{r} \rightarrow \infty} \tilde{\beta} = \lim_{\tilde{r} \rightarrow \infty} \tilde{U}^A = 0, \quad \lim_{\tilde{r} \rightarrow \infty} \tilde{r}^{-1}\tilde{V} = 1 \quad \text{and} \quad \lim_{\tilde{r} \rightarrow \infty} \tilde{h}_{AB} = q_{AB}. \quad (4.9)$$

When solved, the main equations yield

$$\tilde{\beta} = -\frac{1}{32\tilde{r}}\tilde{c}^{AB}\tilde{c}_{AB} - \frac{1}{\tilde{r}^2}\pi\varphi_1^2 + \dots, \quad (4.10a)$$

$$\tilde{U}^A = -\frac{1}{2\tilde{r}^2}\tilde{\partial}_B\tilde{c}^{AB} + \dots, \quad (4.10b)$$

$$\tilde{V} = \tilde{r} - 2\tilde{M} + \dots \quad (4.10c)$$

4.2 Brans-Dicke Theory in the Jordan Frame

As mention in the introduction of this chapter, the asymptotic conditions of the metric are simpler to determine in the Einstein frame, where the scalar field is only minimally coupled to gravity inside the action. However, for physical predictions, it is more relevant to work in the Jordan frame where, though the scalar field now is non-minimally coupled to gravity, the geodesics of the space-time describe the movement of particles. We start by translating the asymptotic conditions to the Jordan frame, and then proceed to repeat the solution of Einstein's equations with the Bondi-Sachs metric, now with the knowledge what to require of our solution. First, we scale our metric by the conformal factor Φ :

$$g_{ab} = \frac{1}{\Phi}\tilde{g}_{ab}, \quad (4.11)$$

where

$$\Phi = \exp(\varphi/\Omega), \quad \Omega = \sqrt{\frac{2\omega + 3}{16\pi}} \quad (4.12)$$

The scalar field is called Φ in this frame, and ω is the Brans-Dicke parameter. General Relativity is recovered in the limit $\omega \rightarrow \infty$, where Φ becomes non-dynamical. The Brans-Dicke action in the Jordan frame is given by [36]

$$S_J^{\text{BD}}[g, \Phi] = \frac{1}{16\pi} \int \left(\Phi R - \frac{\omega}{\Phi} g^{ab} \partial_a \Phi \partial_b \Phi \right) \sqrt{|\det g|} d^4x \quad (4.13)$$

where R is the Ricci scalar in the Jordan frame. The field equations are given by

$$0 = E_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} - 8\pi T_{ab}^{\text{eff}}, \quad (4.14)$$

$$0 = \square\Phi, \quad (4.15)$$

where the stress-energy tensor of the scalar field is given by

$$T_{ab}^{\text{eff}} = \frac{1}{\Phi} (\nabla_a \nabla_b \Phi - g_{ab} \square\Phi) + \frac{\omega}{8\pi\Phi^2} \left(\partial_a \Phi \partial_b \Phi - \frac{1}{2}g_{ab}g^{cd} \partial_c \Phi \partial_d \Phi \right) \quad (4.16)$$

The transformation of the scalar field given by Equation (4.12) can be expanded in powers of r^{-1} as

$$\Phi(\tilde{u}, \tilde{r}, \tilde{x}^A) = \exp\left(\frac{\varphi_0}{\Omega}\right) \left(1 + \frac{\varphi_1}{\Omega} \frac{1}{\tilde{r}} \right) + \mathcal{O}(\tilde{r}^{-2}). \quad (4.17)$$

By itself, a conformal transformation is a mapping of the metric, and it does not imply a change in coordinates; however, in the Jordan frame, the previously

imposed determinant condition clearly no longer holds: $\det g_{AB} = \tilde{r}^4 \Phi^{-2} \det q$, i.e. $\tilde{r}$ is not an areal radius in the Jordan frame. This justifies our decision to introduce separate names for the coordinates in the Einstein frame, as an additional coordinate transformation of the form

$$u = \frac{\tilde{u}}{\sqrt{\Phi_0}}, \quad r = \frac{\tilde{r}}{\sqrt{\Phi}}, \quad x^A = \tilde{x}^A, \quad (4.18)$$

is required for the character of the metric to be enforced in the Jordan frame. The constant term in the expansion of Φ is given by $\Phi_0 = \exp(\varphi_0/\Omega)$. The retarded time also has to be rescaled, such that $-g_{ur} \rightarrow 1$ as $r \rightarrow \infty$. The new metric takes the Bondi-Sachs form,

$$g_{ab} dx^a dx^b = -\frac{V}{r} e^{2\beta} du^2 - 2e^{2\beta} du dr + r^2 h_{AB} (dx^A - U^A du) (dx^B - U^B du), \quad (4.19)$$

with the Bondi gauge conditions

$$g_{rA} = g_{rr} = 0, \quad \det g. \quad (4.20)$$

The functions β , U^A and V , now of the coordinates $x^a = (u, r, x^A)$ can be found to be of the form

$$\beta = -\frac{1}{2r} \frac{\varphi_1}{\Omega \sqrt{\Phi_0}} + \mathcal{O}(r^{-2}) \quad (4.21a)$$

$$U^A = -\frac{1}{2\sqrt{\Phi_0} r^2} (\partial_B c^{AB} - \partial^A \varphi_1) + \mathcal{O}(r^{-3} \log r) \quad (4.21b)$$

$$V = \left(1 + \frac{\partial_u \varphi_1}{\Omega}\right) + \mathcal{O}(r^0) \quad (4.21c)$$

Thus, asymptotically, the functions of the metric in the Jordan frame are of the form

$$\beta = \mathcal{O}(r^{-1}), \quad V = \mathcal{O}(r), \quad U^A = \mathcal{O}(r^{-2}). \quad (4.22)$$

Knowing the asymptotic conditions for the functions of the metric, it is easier to derive their explicit values by repeating the solution of the equations of motion, now in the Jordan frame. We know that the scalar field can be expanded as a power series in r^{-1} with a constant coefficient Φ_0 , i.e.

$$\Phi(u, r, x^A) = \Phi_0 + \frac{\Phi_1(u, x^A)}{r} + \frac{\Phi_2(u, x^A)}{r^2} + \frac{\Phi_3(u, x^A)}{r^3} + \mathcal{O}(r^{-4}), \quad (4.23)$$

The coefficient Φ_1 is connected to the additional polarization of gravitational waves in Brans-Dicke theory, while Φ_0 is related to the effective gravitational constant [36],

$$G(\Phi) = \frac{4 + 2\omega}{3 + 2\omega} \frac{1}{\Phi}. \quad (4.24)$$

For $r \rightarrow \infty$, assuming $G \rightarrow 1$ we have

$$\Phi_0 = \frac{3 + 2\omega}{4 + 2\omega}. \quad (4.25)$$

The procedure to solve the equations of motion follows the same algorithm previously explained for GR and also used in the Einstein frame, the explicit equations can be found in [36]. Here, we restrict ourselves to presenting the results,

$$\beta = -\frac{\Phi_1}{2\Phi_0 r} - \frac{1}{r^2} \left(\frac{1}{32} c^{AB} c_{AB} + \frac{\omega-1}{8\Phi_0^2} \Phi_1^2 + \frac{3\Phi_2}{4\Phi_0} \right) + \mathcal{O}(r^{-3}), \quad (4.26a)$$

$$\begin{aligned} U^A = & -\frac{1}{2r^2} \left(\partial_F c^{AF} - \frac{\partial^A}{\Phi_0} \right) \\ & + \frac{1}{3r^3} \left[c^{AD} \partial^F c_{DF} - \frac{1}{\Phi_0} c^{AD} \partial_D \Phi_1 + \frac{\Phi_1}{\Phi_0} \partial_B c^{AB} - \frac{\Phi_1}{\Phi_0^2} \partial^A \Phi_1 + 6L^A \right] \\ & + \mathcal{O}(r^{-4}), \end{aligned} \quad (4.26b)$$

$$V = \left(1 + \frac{\partial_u \Phi_1}{\Phi_0} \right) r - 2M + \mathcal{O}(r^{-1}) \quad (4.26c)$$

$$h_{AB} = q_{AB} + \frac{c_{AB}}{r} + \mathcal{O}(r^{-2}). \quad (4.26d)$$

As mentioned above, for scalar-tensor theories we also have the additional evolution equation for the scalar field. The component $\partial_u \Phi_1 = N$ is an arbitrary function of integration, while some of the other components are given by

$$\partial_u \Phi_2 = -\frac{1}{2} \mathfrak{D}^2 \Phi_1 \quad (4.27a)$$

$$\begin{aligned} \partial_u \Phi_3 = & -\frac{1}{2\Phi_0} \partial_u \Phi_1 \Phi_2 + \frac{1}{2} M \Phi_1 - \frac{1}{4} (\mathfrak{D}^2 + 2) \Phi_2 - \frac{1}{8\Phi_0} \Phi_1 \mathfrak{D}^2 \Phi_1 \\ & + \frac{1}{4} c^{AB} \partial_A \partial_B \Phi_1 + \frac{1}{8} \Phi_1 \partial_A \partial_B c^{AB} + \frac{1}{2} (\partial_B c^{AB}) (\partial_A \Phi_1) \end{aligned} \quad (4.27b)$$

4.3 Mass-Loss Formulae

The supplementary conditions yield, like in GR, the mass and angular momentum loss formulae. In the case of Brans-Dicke, they are respectively

$$\partial_u \mathcal{M} = -\frac{1}{8} N_{AB} N^{AB} + \frac{1}{4} \partial_A \partial_B N^{AB} - (3 + 2\omega) \frac{1}{4\Phi_0^2} (\partial_u \Phi_1)^2 + \frac{1}{4\Phi_0} \partial_u \mathfrak{D}^2 \Phi_1, \quad (4.28)$$

and

$$\begin{aligned} -3 \partial_u L_A = & \partial_A \mathcal{M} - \frac{1}{4} \partial^E (\partial_E \partial^F c_{AF} - \partial_A \partial^F c_{EF}) + \frac{1}{16} \partial_A (c_{EF} N^{EF}) \\ & - \frac{1}{2} \partial_C c^{CF} N^{AF} + \frac{1}{4} c^{EF} (\partial_A N_{EF}) + \frac{1}{8\Phi_0} \partial_A \mathfrak{D}^2 \Phi_1 \\ & - \frac{1}{4\Phi_0^2} (2 + 3\omega) (\partial_A \Phi_1) (\partial_u \Phi_1) + \frac{\Phi_1}{4\Phi_0^2} (4 + \omega) \partial_A \partial_u \Phi_1 \\ & + \frac{1}{4\Phi_0} \partial_u (c_{AB} \partial^B \Phi_1 - \Phi_1 \partial^B c_{AB}). \end{aligned} \quad (4.29)$$

The mass aspect $\mathcal{M}$ as been redefined as

$$\mathcal{M}(u, x^A) = M(u, x^A) - \frac{1}{4\Phi_0^2} \Phi_1 \partial_u \Phi_1, \quad (4.30)$$

such that $\mathcal{M}$ is a strictly decreasing function, more closely analogous to the Bondi mass aspect in General Relativity [36].

By using the developed Maxima software, these results may also be reproduced for the choice of axial symmetric coordinates. This yields

$$\begin{aligned} M_u = & -c_u^2 + \frac{1}{2} (c_{\theta\theta} + 3c_\theta \cot \theta - c)_u \\ & + \frac{1}{4\Phi_0} (D_\theta \cot \theta + D_{\theta\theta})_u + \frac{1}{4\Phi_0^2} (DD_{uu} - 2D_u^2\omega - 2D_u^2) \end{aligned} \quad (4.31)$$

and

$$\begin{aligned} -3N_u = & M_\theta + 3cc_{u\theta} + 4cc_u \cot \theta + c_u c_\theta \\ & + \frac{1}{8\Phi_0} (D_\theta \cot \theta + D_{\theta\theta})_\theta + \frac{1}{\Phi_0} (2D_\theta c - Dc_\theta - 2Dc \cot \theta)_u \\ & + \frac{1}{4\Phi_0^2} (5DD_{u\theta} + DD_{u\theta}\omega - 3D_u D_\theta \omega - D_u D_\theta) \end{aligned} \quad (4.32)$$

This last result may not seem very useful *per se*. Clearly, one could just substitute the metric h_{AB} for the Bondi metric in Equations (4.28) and (4.29). However, its significance lies in the fact that it was calculated using our Maxima software instead; this provides a good way to test the results of the program with known literature, other than the original Bondi, van der Burg and Metzner publication [10]. Furthermore, once the program was finally working—which did take a considerable amount of work—the mass-loss formula could be reproduced in less than ten minutes. It is very likely that the calculations in [36] were also carried out using computer algebra systems, but the source code was not made available together with the publications—as far as we are aware.

Chapter 5

Bocharova-Bronnikov-Melnikov-Bekenstein Theory

In this chapter, we consider the case of BBMB theory, a scalar-tensor theory based on a conformal invariant scalar equation [8, 3].

This theory is of great relevance in this work; while it still yields relatively simple field equations, compared with more exotic theories of gravity, it provides a case, in which—as far as we are aware—no previous attempt has been made to apply and study the Bondi-Sachs formalism. Here, we have the possibility to test the usefulness of our software by producing new results. Because of this, and the limitations of our software already mentioned above, this chapter only provides solutions to the field equations and mass-loss formula in axial symmetry, using the Bondi metric [9].

In Section 5.1, we present the equations of motion in their original form, and proceed to simplify them to improve the performance of our software. The field equations are then solved in Section 5.2, and the loss formulae of the theory are listed in 5.3.

5.1 Introduction

In the literature it is known that Maxwell’s equations, the massless Dirac equation, and the two-component neutrino equation are all conformal invariant [44, 30]; however, the massless scalar equation, as we have seen for example in Brans-Dicke theory, i.e.

$$\square\Phi = 0, \tag{5.1}$$

is not. BBMB theory considers instead the scale invariant scalar equation given by [3]

$$\square\Phi - \frac{1}{6}R\Phi = 0. \tag{5.2}$$

BBMB theory has been of interest in the literature as it provides an answer to two existing questions regarding asymptotically flat black holes with scalar hair [17]:

- It provides a scalar-hairy black holes that does not reduce to Schwarzschild black hole.

- Has a diverging scalar field at the black hole horizon that does not translate into a curvature singularity at the (would be) horizon.

The action for BBMB theory can be expressed as [3]

$$S^{\text{BBMB}}[g, \Phi] = \int \left(\frac{R}{16\pi} - \frac{1}{2} g^{ab} \partial_a \Phi \partial_b \Phi - \frac{1}{12} R \Phi^2 \right) \sqrt{|\det g|} \, d^4 x, \quad (5.3)$$

where Φ is the scalar field and R is the Ricci scalar. Though not usually considered as such in the literature, in our language, this action can be seen as an action in Jordan frame with the respective $A(\Phi)$ in equation (3.7) given by

$$A(\Phi) = 1 - \frac{4\pi}{3} \Phi^2. \quad (5.4)$$

The field equations are

$$0 = E_{ab} = R_{ab} - \frac{1}{2} R g_{ab} - 8\pi \tilde{T}_{ab}^{\text{eff}}, \quad (5.5)$$

$$0 = \square \Phi - \frac{1}{6} \Phi R, \quad (5.6)$$

where the effective energy-momentum tensor is given by

$$T_{ab}^{\text{eff}} = \frac{1}{4\pi} \left[\partial_a \Phi \partial_b \Phi - \frac{1}{2} g_{ab} g^{cd} \partial_c \Phi \partial_d \Phi - \frac{1}{6} (g_{ab} \square - \nabla_a \nabla_b + G_{ab}) \Phi^2 \right]. \quad (5.7)$$

However, before using the equations in their current form inside Maxima, it is useful to reduce the complexity of the computation by noting that by taking the trace of (5.5), one obtains

$$\begin{aligned} -R &= 2 \left(-g^{ab} \partial_a \Phi \partial_b \Phi + \frac{1}{2} \square \Phi^2 - \frac{1}{6} R \Phi^2 \right) \\ &= 2\Phi \left(\square \Phi - \frac{1}{6} R \Phi \right), \end{aligned} \quad (5.8)$$

which due to equation (5.6) implies that the Ricci scalar vanishes [7]. Hence,

$$R = 0, \quad \text{and} \quad \square \Phi = 0. \quad (5.9)$$

Additionally, we rescale the scalar field as

$$\Phi = \sqrt{3} \zeta, \quad (5.10)$$

such that the field equations take now the form

$$0 = E_{ab} = R_{ab} - \frac{1}{1 - \zeta^2} (4 \partial_a \zeta \partial_b \zeta - g_{ab} g^{cd} \partial_c \zeta \partial_d \zeta - 2 \zeta \nabla_a \nabla_b \zeta), \quad (5.11a)$$

$$0 = \square \zeta, \quad (5.11b)$$

which we can more easily handle inside Maxima.

5.2 Solution to the Field Equations

Now that the field equations have been simplified, we proceed to solve them for the Bondi metric (2.52). In the preceding chapter, following the steps of previous publications by other authors [36], we first solved the equations of motion in the Einstein frame in order to determine the asymptotic conditions for the metric functions. However, we argue that this is not really necessary to repeat for BBMB theory, or even in general.

It was already known that β could—in principle—contain up to a term of order r^0 , which for General Relativity could be eliminated through a properly selected coordinate transformation (see H function in [10]).

We claim that once the Bondi gauge condition has been imposed, the solution of the equations of motion using the most general possible ansatz allowed by the asymptotic conditions (2.28), including the possibility of terms β_0 and β_1 (see Equation (2.29a)), will either succeed, or lead to an evident contradiction in the equations for the expansion terms.

This idea to skip the analysis in the Einstein frame was inspired by noting, that once the form of the metric's functions was determined, these results were no longer used for the solution of the Bondi-Sachs metric in the Jordan frame; the metric's functions were determined “from scratch”. Hence, with the help of the our program, we can effectively attempt a “brute force” solution, and determine in little time if this lead to a contradiction.

Nevertheless, though we just provided an argument against the necessity of the step, we still briefly consider the form of the scalar field's transformation into the Einstein frame. First, we recall our comment on conformal transformations, and conclude that BBMB in the Einstein frame has the same action as Brans-Dicke theory. The differential equation for the transformation of the scalar field is given by

$$\frac{d\varphi}{d\zeta} = \pm \frac{1}{1 - \kappa\zeta^2/6} \sqrt{1 + \kappa(12\kappa - 1/6)\zeta^2} \sim \pm \frac{1}{\zeta}, \quad \kappa = 8\pi. \quad (5.12)$$

This equation can be solved analytically, but this would be of little use for our purposes. It suffices to note that

$$\zeta \sim \exp(\pm\varphi). \quad (5.13)$$

Thus, we can expect, similar to Brans-Dicke theory, that after the necessary coordinate transformations required to keep the Bondi gauge after transforming the Einstein frame expansion into the Jordan frame, β will gain a $1/r$ term—see derivation of equations (4.21a)–(4.21c).

For the scalar field we choose the ansatz

$$\zeta = \zeta_0(u, \theta) + \zeta_1(u, \theta)r^{-1} + \zeta_2(u, \theta)r^{-2} + \dots \quad (5.14)$$

For γ , β , U and V we assume the usual expansions presented on (2.29a)–(2.29d) and adjusted for axial symmetry. Solving the equations of motion for the metric

yields

$$\gamma = c(u, \theta)r^{-1} + \left[C(u, \theta) - \frac{1}{6}c^3 \right] + \dots \quad (5.15a)$$

$$\beta = \zeta_0 \zeta_1 r^{-1} - \frac{1}{4} (c^2 - 6\zeta_0 \zeta_2 + 2\zeta_0^2 \zeta_1^2) r^{-2} + \dots \quad (5.15b)$$

$$U = -[2c \cot \theta + c_\theta + \zeta_0 (\zeta_1)_\theta] r^{-2} + [2N(u, \theta) + 3cc_\theta + 4c^2 \cot \theta] r^{-3} + \dots \quad (5.15c)$$

$$V = [1 - 2\zeta_0 (\zeta_1)_u] r - 2M(u, \theta) + \dots \quad (5.15d)$$

From the evolution equation for the scalar field, the first two terms imply

$$0 = 2(\zeta_0)_u \quad (1/r), \quad (5.16a)$$

$$0 = (\zeta_0)_\theta + (\zeta_0)_{\theta\theta} + 4\zeta_0 \zeta_1 (\zeta_0)_u \quad (1/r^2). \quad (5.16b)$$

Thus, once again, the first coefficient of the scalar field, i.e. ζ_0 , is a constant. The complete time evolution of ζ is given by

$$(\zeta_1)_u = D_u(u, \theta) \quad (5.17a)$$

$$(\zeta_2)_u = -\frac{1}{2} (D_\theta \cot \theta + D_{\theta\theta}) \quad (5.17b)$$

$$(\zeta_2)_u = \frac{1}{4} [(3Dc_\theta + 6D_\theta c - \zeta_0 DD_\theta - \zeta_\theta) \cot \theta + Dc_{\theta\theta} + 4D_\theta c_\theta + 2D_{\theta\theta} c - 2Dc + 2DM - \zeta DD_{\theta\theta} + 4\zeta_0 \zeta_2 D_u - (\zeta_2)_{\theta\theta} - 2\zeta_2] \quad (5.17c)$$

...

5.3 Loss Formulae

With the solution to the equations of motion, we now use the supplementary conditions to derive the mass-loss and momentum-loss formulae for the BBMB theory. These are given by

$$M_u = -c_u^2 + \frac{1}{2} (3c_\theta \cot \theta + c_{\theta\theta} - c)_u + (1 - \zeta_0^2) DD_{uu} - 2(1 + \zeta_0^2) D_u^2, \quad (5.18)$$

and

$$\begin{aligned} -3N_u &= M_\theta + 3cc_{u\theta} + 4cc_u \cot \theta + c_u c_\theta \\ &+ \zeta_0^2 (D_u D_\theta + 3DD_{u\theta}) + 2\zeta_0 (D_u c_\theta)_u + 2DD_{u\theta} - 4D_u D_\theta \\ &- \frac{1}{4} \zeta_0 (D_\theta \cot \theta + D_{\theta\theta}) + 4\zeta_0 (Dc \cot \theta - D_\theta c)_u. \end{aligned} \quad (5.19)$$

The aspect functions are yet to be redefined to a non-increasing quantity. This has been left out of the scope of this thesis.

With the results of the previous section and loss formulae just derived, a possible next step could be to derive the asymptotic symmetries for the BBMB theory, and determine if there any changes to their structure when compared

to General Relativity or Brans-Dicke theory. However, this step is best taken when a version of the program, capable of handling a general h_{AB} metric, is developed. Instead, we close this chapter at this point and proceed with a case where the potential in the Bergmann metric [4, 31] is non-vanishing.

Chapter 6

Theories with no Asymptotically Flat Spacetimes and Other Challenges

In this chapter, we explore the challenges one may face when applying the software developed in the context of this thesis to more complicated scalar-tensor theories of gravity. In Section 6.1, we first postulate a conjecture to determine if a given scalar-tensor theory of gravity possesses an asymptotically flat spacetime solution. This idea is tested in Section 6.2 for the case of Starobinsky gravity [35], where it is shown that using the methods presented in this work no non-trivial asymptotically flat spacetime can be found. Finally, Section 6.3 shows further complications that one may encounter when using our software in its current state, and that need to be covered by future updates.

6.1 Introduction

The topic of asymptotic flatness in GR is a complicated one. As mentioned in previous chapters, there appears to be no consensus about the form of the negative series expansion of the functions of the Bondi-Sachs metric for scalar-tensor theories of gravity. However, we were able to avoid this issue for the BBDB theory by a “brute force” expansion.

In general, the requirements for the metric of a spacetime to be asymptotically flat can be translated into requiring the metric to be expressible in the Bondi-Sachs form [37, 41]. For matters of simplicity, in this chapter we make the *conjecture* that contrapositive is also valid for scalar-tensor theories of gravity. That is, our conjecture states the follows:

If a scalar-tensor theory does not have a spacetime solution of the field equations with a metric g_{ab} and scalar field Φ , such that through an appropriate choice of coordinates the metric g_{ab} has the Bondi-Sachs form and both can be expanded in negative powers of the

inverse areal, lightlike coordinate r , then said theory has no asymptotically flat spacetime, and a mass-loss formula, with the properties considered in this work, is not computable.

This is undoubtedly a very bold claim, for which we unfortunately provide no proof in this thesis. This is instead, an intuition that developed while attempting, without success, to apply the Bondi-Sachs formalism to scalar-tensor theories of the form (3.7) with a non-vanishing potential; in particular, Starobinsky gravity, as we shall see in the following section.

6.2 Starobinsky Gravity

We start this chapter by introducing the theory of Starobinsky gravity. A theory proposed as a simple model for inflation in the early universe [35]. It is a member of the so called $f(R)$ -theories of gravity, characterized by the action

$$S[g] = \frac{1}{16\pi} \int f(R) \sqrt{|\det g|} \, d^4x, \quad (6.1)$$

where the R term in the Einstein-Hilbert action is replaced by a function of the scalar curvature. The function f is commonly chosen to be analytic such that it allows for a power series expansion. In particular, for Starobinsky inflation, we have $f(R) = R + \lambda R^2$ and the action takes the concrete form [35, 40]

$$S[g] = \frac{1}{16\pi} \int (R + \lambda R^2) \sqrt{|\det g|} \, d^4x, \quad (6.2)$$

where λ is some constant.

Heuristically, one can think of the square term as dominating the action for large curvature; thus, also for great matter concentration. This creates a period of accelerated expansion, until matter is effectively diluted and the theory reduces to the well known case of GR. A more rigorous discussion is out of the scope of this text. For our purposes, the interest for this theory lies in the possibility to reformulate it in a scalar-tensor form, as it provides a concrete example of a Bergman-Wagoner theory of the form (3.7) with a non-zero potential. In particular, it exemplifies the possible issues encountered with theories for which a mass-loss is unlikely to exist. An algorithm for determining the existence of non-trivial, asymptotically flat spacetimes in a theory may be formulated.

To derive the scalar-tensor form of Starobinsky, we first introduce a potential $\Pi(\Phi)$, defined to be the Legendre transform of $f(R)$. Thus,

$$\Pi(\Phi) = F(R(\Phi), \Phi) = \Phi R - f(R),$$

such that the function F is extremal, i.e. $\partial F / \partial R = 0$. Therefore,

$$\Phi = \frac{df}{dR} = 1 + 2\lambda R \quad \text{and} \quad R = \frac{\Phi - 1}{2\lambda}.$$

Replacing in the definition of V we have

$$\Pi = \frac{(\Phi - 1)^2}{4\lambda} \quad \text{and} \quad \frac{d\Pi}{d\Phi} = \frac{\Phi - 1}{2\lambda} = R. \quad (6.3)$$

Finally, from (6.2) we have

$$f(R) = \Phi R - F(R, \Phi) = \Phi R - \Pi(\Phi), \quad (6.4)$$

and we can rewrite the action of Starobinsky gravity equivalently in the Jordan frame as

$$S_J[g, \Phi] = \frac{1}{16\pi} \int [\Phi R - \Pi(\Phi)] \sqrt{|\det g|} \, d^4x. \quad (6.5)$$

According to the procedure we have seen thus far, we would now proceed to derive the equations of motion, identify the the main equations, solve first the hypersurface equations, and then continue by solving the evolution equations. However, by looking before at the evolution equation for the scalar field (see Appendix B), given by

$$\square \Phi = \frac{16\pi}{3} \left(\Phi \frac{dV}{d\Phi} - 2\Pi \right), \quad (6.6)$$

with the coefficients for the functions not yet solved for, we notice the following results must hold

$$\Phi_0 = 1, \quad (6.7a)$$

$$\Phi_1 = \frac{3\lambda}{4\pi} (\Phi_0)_u, \quad (6.7b)$$

$$\Phi_2 = -\frac{3\lambda}{8\pi} [(\Phi_0)_\theta \cot \theta + (\Phi_0)_{\theta\theta} - U_1(\Phi_0)_\theta + 4\beta_1(\Phi_0)_u]. \quad (6.7c)$$

$$\dots \quad (6.7d)$$

This chain of implications is such that the scalar field must vanish. Therefore, the only spacetime left is just equivalent to GR.

6.3 Einstein-Gauss-Bonnet-dilaton

We now close this chapter by presenting the Einstein-Gauss-Bonnet-dilaton theory. Though we were not able to produce any results at the time of writing, the scale of the task, and the challenges faced, show the significance and utility of our software package; as well as the issues left yet to resolve.

In four dimensions, the Einstein-Gauss-Bonnet-dilaton action is given by [6, 23]

$$S^{\text{EGBd}}[g, \Phi] = \frac{1}{16\pi} \int \left(R - \frac{1}{2} g^{ab} \partial_a \Phi \partial_b \Phi + \frac{1}{4} \alpha e^{\gamma \Phi} R_{\text{GB}}^2 \right) \sqrt{|\det g|} \, d^4x, \quad (6.8)$$

where the Gauss-Bonnet term R_{GB}^2 in four dimensions is defined as

$$R_{\text{GB}}^2 = R_{abcd} R^{abcd} - 4R_{ab} R^{ab} + R^2, \quad (6.9)$$

and is coupled to the dilaton via the coupling constant α and the dilaton coupling constant γ .

The presence of a scalar field in the theory is required. The Gauss-Bonnet term by itself is a total derivative in four dimensions, and would otherwise not influence the dynamics of the system [1].

The field equations of the metric are given by [6]

$$G_{ab} = \frac{1}{2} \left(\partial_a \Phi \partial_b \Phi - \frac{1}{2} g_{ab} g^{cd} \partial_c \Phi \partial_d \Phi \right) - \frac{1}{4} \alpha e^{\gamma \Phi} (H_{ab} + 4 (\partial^a \Phi \partial^b \Phi + \partial^\rho \partial^\sigma \Phi)), \quad (6.10)$$

where the auxiliary tensors H_{ab} and P_{abcd} are given by

$$H_{ab} = 2 (R R_{ab} - 2 R_{ac} R^c_b - 2 R_{acbd} R^{cd} + R_{acde} R_b^{cde}) - \frac{1}{2} g_{ab} R_{GB}^2, \quad (6.11a)$$

$$P_{abcd} = R_{abcd} + g_{ad} R_{cb} - g_{ac} R_{db} + g_{bc} R_{da} - g_{bd} R_{ca} + \frac{1}{2} R (g_{ac} g_{db} - g_{ad} g_{cb}). \quad (6.11b)$$

The dilaton field equations are given by

$$\square \Phi = \frac{1}{4} \alpha \gamma e^{\gamma \Phi} R_{GB}^2. \quad (6.12)$$

At this point, the procedure to apply the Bondi-Sachs formalism is clear. Solve the main equations equivalent to (2.17) by solving first the hypersurface equations (2.23), and then evolving the solution through the evolution equations (2.25), in addition to the evolution equation due to the scalar field (6.12).

However, EGBd theory presents complications not yet handled by our software package. In the interest of further development of our software, we documented these issues in the following text.

Firstly, the Bondi gauge, specifically the condition $g_{rr} = 0$, guarantees for simpler theories, where at most some kinetic term is present, that the equation E_r^u yields differential equation involving only β . Similarly, the rest of the hypersurface equations also follow this hierarchy: solving any of them depends only on the results collected up to that point from the previous hypersurface equations. This is no longer the case for more complicated theories, where terms like the Gauss-Bonnet term result in a coupled system of equations. For example, the first order term of the first hypersurface equation, i.e. Equation (2.24a), is

$$0 = 2\beta_1 (\Phi_0)_{\theta\theta} + U_1 (\Phi_0)_{u\theta} \quad (1/r). \quad (6.13)$$

Though by no means an insurmountable obstacle, our code is still being developed, and this case needs still to be handled.

The second challenge shows at the same time how useful a package, such as ours, really is. The equation we have just considered, in its full expanded form, is made up of 1017 distinct terms. Some clever observation may eliminate the need to evaluate all of them, for example, for the Kretschmann scalar is known that

$$R_{abcd} R^{abcd} \sim r^{-6}. \quad (6.14)$$

Thus, when calculating the loss formulae, which are known to be encoded exclusively in a r^{-2} term (see Equation (2.22)), one can omit many of the resulting terms. Nevertheless, it is clear that having a program capable of carrying out a mostly algorithmic procedure is of great usefulness, specially for reproducible results in a comparable short amount of time, when compare with the effort of manual calculation.

Finally, although great effort was made to optimize the calculations, as seen by the fact that the original results from Bondi, van der Burg and Metzner [10] are reproducible in less than 10 minutes on a personal laptop, the large amount of terms in more complicated theories still results in long compute times: the expansion of the 1017 terms for the first hypersurface equation in EGBd theory required one day on a 8-core system. Parallelization was improvised by splitting terms between multiple Maxima instances using scripts and the help of GNU Parallel [38] (see code repository); however, a more robust way to calculate terms in parallel is clearly needed.

Chapter 7

Summary

7.1 Bondi

$$M_u = -c_u^2 + \frac{1}{2} (c_{\theta\theta} + 3c_\theta \cot \theta - c)_u \quad (7.1a)$$

$$-3N_u = M_\theta + 3cc_{u\theta} + 4cc_u \cot \theta + c_u c_\theta \quad (7.1b)$$

7.2 Brans-Dicke

$$M_u = -c_u^2 + \frac{1}{2} (c_{\theta\theta} + 3c_\theta \cot \theta - c)_u \quad (7.2a)$$
$$+ \frac{1}{4\Phi_0} (D_\theta \cot \theta + D_{\theta\theta})_u + \frac{1}{4\Phi_0^2} (DD_{uu} - 2D_u^2 \omega - 2D_u^2)$$

$$-3N_u = M_\theta + 3cc_{u\theta} + 4cc_u \cot \theta + c_u c_\theta$$
$$+ \frac{1}{8\Phi_0} (D_\theta \cot \theta + D_{\theta\theta})_\theta + \frac{1}{\Phi_0} (2D_\theta c - Dc_\theta - 2Dc \cot \theta)_u \quad (7.2b)$$
$$+ \frac{1}{4\Phi_0^2} (5DD_{u\theta} + DD_{u\theta}\omega - 3D_u D_\theta \omega - D_u D_\theta)$$

7.3 BBMD

$$M_u = -c_u^2 + \frac{1}{2} (3c_\theta \cot \theta + c_{\theta\theta} - c)_u \quad (7.3a)$$
$$+ (1 - \zeta_0^2) DD_{uu} - 2(1 + \zeta_0^2) D_u^2,$$

$$-3N_u = M_\theta + 3cc_{u\theta} + 4cc_u \cot \theta + c_u c_\theta$$
$$+ \zeta_0^2 (D_u D_\theta + 3DD_{u\theta}) + 2\zeta_0 (D_u c_\theta)_u + 2DD_{u\theta} - 4D_u D_\theta \quad (7.3b)$$
$$- \frac{1}{4} \zeta_0 (D_\theta \cot \theta + D_{\theta\theta}) + 4\zeta_0 (Dc \cot \theta - D_\theta c)_u.$$

Chapter 8

Outlook and Conclusion

In this thesis, starting in Chapter 2, we have provided an overview of the key ideas and construction of Bondi-Sachs formalism, specially in its generalized form, in order to explain how it could be generalized to alternative theories of gravity, in particular, scalar-tensor theories derived from the Bergman-Wagoner action. In Chapter 3, we covered the fundamental concepts of conformal transformations and the usual conformal frames found in the literature.

Our final goal is to reproduce the calculations of the Bondi-Sachs formalism for possibly many alternative gravity theories in a semi-automatic, or if possible, even automatic way, using a custom written computer algebra package for Maxima, a libre CAS with a long development history. However, the scope of this thesis had to be adjusted to account for the current limitations due to no complete tensor package being available for Maxima. Thus, results produced with our package were restricted to only the original axial symmetric Bondi metric.

As a starting point, in Chapter 4, we reviewed and reproduced the existing results in the literature, where the Bondi-Sachs formalism is applied to Brans-Dicke theory, one of the simplest cases of the Bergman-Wagoner action. We explained the derivation of the asymptotic conditions for the metric in the Jordan frame, starting from general considerations in the Einstein frame, and translating the metric via the necessary conformal transformation. However, after the conditions were determined, the equations of motion were simply solved again in the Jordan frame. We closed the chapter by comparing the results in the literature, expressed in a general sub-metric h_{AB} , with the results produced by our Maxima package. This provided a good test for the package.

Chapter 5 introduced the Bocharova-Bronnikov-Melnikov-Bekenstein action, selected specially for its simplicity and similarity with the Brans-Dicke action (from the point of view of the complexity for our software). The BBMB theory presented the first test scenario for our Maxima package to produce novel results, albeit likewise limited to axial symmetry. We solved the equations of motion and provided both mass-loss and impulse-loss formulae.

Finally, in Chapter 6, we applied the Bondi-Sachs formalism to the case of Starobinsky gravity, which although usually considered a $f(R)$ theory, it is easily converted to a scalar-tensor theory by an appropriate Legendre transformation. We briefly covered the Einstein-Gauss-Bonnet-dilaton theory, as case of existing interest in the community, which unfortunately was not able to be

solved during the preparation of this theses, due to challenges of coupling in the main equations. However, this problem was purely a topic of time availability, we currently have no reason to believe that EGBd gravity couldn't be solved using our software together with some manual intervention.

Work on this topic will continue after the submission of this thesis. There exist interest to start development of a Maxima package for general tensor manipulations, which would allow for working with a general h_{AB} metric. Thus, once this is available, more general results will be computed to expand on the work presented in this work.

As mentioned, there exists the motivation for applying the Bondi-Sachs formalism to EGBd theory. This is also a goal we intend to pursue.

There also still exists the open question on how to best approach the issue asymptotic flatness for scalar-tensor theories of gravity. The issue is of great relevance, specially given that we intend to employ our software for theories where the notion of conformal frames is not particularly useful, i.e. no possibility to transform to a frame, where the equations of motion have a known form like Einstein-Klein-Gordon. As explained in Chapter 2, the Bondi formalism is heavily dependent on the effective stress-energy tensor being both symmetric and divergence free. This could pose difficulties with more exotic alternative theories of gravity, where this may no longer be case. The procedure is also heavily dependent on the causal structure of General Relativity, e.g. the expectation of some advantage in using retarded coordinates implicitly assumes that gravitational waves propagate with the speed of light—though there is much experimental evidence supporting this [5], it may prove interesting to consider theories where this is not strictly the case. Some more fundamental generalizations may need to be introduced into the theory, in order to cover all these issues.

Though our work was mainly focused on the derivation of loss formulae, partially due to them being relatively direct calculations, even for the theories we considered, i.e. BBMB and Starobinsky, there remains the task to calculate the asymptotic symmetry group, also known as BMS group. It is in principle also possible to make explicit calculations of the loss functions, and to estimate the flow of energy leaving an isolated gravitational system. Potentially, this could be compared with observations.

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Appendix A

Weyl Metric in Polar Coordinates

The following derivation is a result long already known in the literature. However, the listed sources did not provide a derivation, which at the time of writing this thesis was thought to be needed for proving some results.

Expressed in polar coordinates, the Weyl metric is given by

$$g_{ab}^{(W)} dx^a dx^b = -e^{2\psi} dt^2 + e^{2\sigma-2\psi} (dR^2 + R^2 d\Theta^2) + e^{-2\psi} R^2 \sin^2(\Theta) d\phi^2. \quad (\text{A.1})$$

The Ricci tensor trivially fulfills

$$R_{01} = R_{02} = R_{03} = R_{13} = R_{23} = 0.$$

We also have one redundant equation:

$$e^{-4\psi} R^2 \sin^2 \Theta R_{00} = R_{33}.$$

Thus, Einstein's field equations take the form

$$0 = e^{2\sigma-4\psi} R^2 R_{00} = \psi_{\Theta} \cot \Theta + \psi_{\Theta\Theta} + R^2 \psi_{RR} + 2R\psi_R, \quad (\text{A.2a})$$

$$0 = R^2 (e^{2\sigma-4\psi} R_{00} - R_{11}) = \sigma_{\Theta} \cot \Theta + 2R^2 \psi_R^2 + \sigma_{\Theta\Theta} + R^2 \sigma_{RR}, \quad (\text{A.2b})$$

$$0 = (R_{22} - R^2 R_{11})/2 = \sigma_{\Theta} \cot \Theta + R^2 \psi_R^2 - \psi_{\Theta}^2 - R\sigma_R, \quad (\text{A.2c})$$

$$0 = RR_{12} = R\sigma_R \cot \Theta - 2R\psi_R \psi_{\Theta} + \sigma_{\Theta}. \quad (\text{A.2d})$$

Equation (A.2a) is similar to the Laplace's equation in spherical coordinates. We impose the boundary condition $\psi \rightarrow 0$ as $r \rightarrow \infty$, and solve it by separation of variables. This yields

$$\psi(R, \Theta) = - \sum_{n=0}^{\infty} a_n \frac{P_n(\cos \Theta)}{R^{n+1}},$$

where P_n are the Legendre polynomials.

Equations (A.2c) and (A.2d) build a system of equations in σ_R and σ_{Θ} , with the solution

$$\sigma_R = 2\psi_R \psi_{\Theta} \cos \Theta \sin \Theta - (R^{-1} \psi_{\Theta}^2 - R\psi_R^2) \sin^2 \Theta \quad (\text{A.3a})$$

$$\sigma_{\Theta} = 2R\psi_R \psi_{\Theta} \sin^2 \Theta + (\psi_{\Theta}^2 - R^2 \psi_R^2) \cos \Theta \sin \Theta. \quad (\text{A.3b})$$

Plugging the results into (A.2b) yields

$$2(R\psi_R + \psi_\Theta \cot \Theta)(\psi_\Theta \cot \Theta + \psi_{\Theta\Theta} + R^2\psi_{RR} + 2R\psi_R) = 0,$$

which clearly ψ fulfills. Thus, we only need to continue working with the equations for σ_R and σ_Θ . We introduce the notation

$$P_n = P_n(\cos \Theta), \quad (\text{A.4})$$

$$P'_n = P'_n(\cos \Theta), \quad (\text{A.5})$$

and write

$$\psi_R = \sum_{n=0}^{\infty} a_n(n+1)R^{-(n+2)}P_n, \quad (\text{A.6})$$

$$\psi_\Theta = \sum_{n=0}^{\infty} a_n R^{-(n+1)}P'_n \sin \Theta. \quad (\text{A.7})$$

Hence, σ can be derived from equations (A.3), and is given by

$$\begin{aligned} \sigma = - \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} a_n a_l \frac{R^{-(n+l+2)}}{n+l+2} [(n+1)(l+1)P_n P_l \sin^2 \Theta \\ + 2(n+1)P_n P'_l \cos \Theta \sin^2 \Theta - P'_n P'_l \sin^4 \Theta], \end{aligned} \quad (\text{A.8})$$

where the integration constant vanishes due to the boundary conditions. The Legendre polynomials satisfy the recursion rule

$$(x^2 - 1)P'_n(x) = (n+1)(P_{n+1}(x) - xP_n(x)),$$

which can be rewritten as

$$P'_n = (n+1) \csc^2 \Theta (\cos \Theta P_n - P_{n+1}). \quad (\text{A.9})$$

Finally, we use this last result to simplify the equation for σ

$$\sigma(R, \Theta) = - \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} a_n a_l \frac{(n+1)(l+1)}{n+l+2} \frac{P_n P_l - P_{n+1} P_{l+1}}{R^{n+l+2}} \quad (\text{A.10})$$

Appendix B

Starobinsky Equations of Motion in the Jordan Frame

The action of Starobinsky gravity, as a scalar-tensor theory expressed in the Jordan frame, is given by (see Section 6.2)

$$S_J[g, \Phi] = \frac{1}{16\pi} \int [\Phi R - \Pi(\Phi)] \sqrt{|\det g|} d^4x. \quad (\text{B.1})$$

The variation of action (B.1) with respect to the metric yields

$$\begin{aligned} \delta S = \frac{1}{16\pi} \int \bigg\{ & \Phi(\nabla^a v_a) \\ & + \left[\Phi \left(R_{ab} - \frac{1}{2} R g_{ab} \right) + \frac{1}{2} \Pi(\Phi) g_{ab} \right] \delta g^{ab} \bigg\} \sqrt{|\det g|} d^4x, \end{aligned} \quad (\text{B.2})$$

where (see [41])

$$v_a = \nabla^b (\delta g_{ab}) - g^{cd} \nabla_a (\delta g_{cd}) \quad \text{and} \quad \delta g_{ab} = -g_{ac} g_{bd} \delta g^{cd}. \quad (\text{B.3})$$

The first term may be integrated by parts

$$\int \Phi(\nabla^a v_a) \sqrt{|\det g|} d^4x = \int (g_{ab} \square \Phi - \nabla_a \nabla_b \Phi) \delta g^{ab} \sqrt{|\det g|} d^4x. \quad (\text{B.4})$$

Thus, the field equations of the metric are given by

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{1}{\Phi} \left(\nabla_a \nabla_b \Phi - g_{ab} \square \Phi - \frac{1}{2} \Pi g_{ab} \right) \quad (\text{B.5})$$

The field equations for the scalar field can be derived from the Legendre transformation used to derive the Jordan frame action and the contraction of field equations for the metric

$$\frac{d\Pi}{d\Phi} = R = \frac{1}{\Phi} (3\square\Phi + 2\Pi). \quad (\text{B.6})$$

Hence,

$$\square\Phi = \frac{16\pi}{3} \left(\Phi \frac{d\Pi}{d\Phi} - 2\Pi \right). \quad (\text{B.7})$$